May 2012

The linear multiplet and ectoplasm

Daniel Butter, Sergei M. Kuzenko and Joseph Novak

School of Physics M013, The University of Western Australia
35 Stirling Highway, Crawley W.A. 6009, Australia
daniel.butter, sergei.kuzenko, joseph.novak@uwa.edu.au

Abstract

In the framework of the superconformal tensor calculus for 4D $\mathcal{N} = 2$ supergravity, locally supersymmetric actions are often constructed using the linear multiplet. We provide a superform formulation for the linear multiplet and derive the corresponding action functional using the ectoplasm method (also known as the superform approach to the construction of supersymmetric invariants). We propose a new locally supersymmetric action which makes use of a deformed linear multiplet. The novel feature of this multiplet is that it corresponds to the case of a gauged central charge using a one-form potential not annihilated by the central charge (unlike the standard $\mathcal{N} = 2$ vector multiplet). Such a gauge one-form can be chosen to describe a variant nonlinear vector-tensor multiplet. As a byproduct of our construction, we also find a variant realization of the tensor multiplet in supergravity where one of the auxiliaries is replaced by the field strength of a gauge three-form.

1 Introduction

In $\mathcal{N}=2$ Poincaré supersymmetry in four space-time dimensions (4D), the linear multiplet was introduced by Sohnius [1] as a superfield Lagrangian describing the dynamics of matter hypermultiplets coupled to Yang-Mills superfields [2]. Following [1, 3], the linear multiplet is a real isotriplet superfield, $L^{ij}=L^{ji}$ and $\overline{L^{ij}}=L_{ij}:=\varepsilon_{ik}\varepsilon_{jl}L^{kl}$, subject to the constraints

$$D_\alpha^{(i}L^{jk)}=\bar{D}_{\dot{\alpha}}^{(i}L^{jk)}=0\ . \tag{1.1}$$

Here D_α^i and $\bar{D}_{\dot{\alpha}}^i$ are the $\mathcal{N}=2$ spinor covariant derivatives with a real central charge Δ. The action proposed in [1] has the form

$$S=-\frac{1}{12}\int \mathrm{d}^4x\,\Big(D^{\alpha(i}D_\alpha^{j)}+\bar{D}_{\dot{\alpha}}^{(i}\bar{D}^{j)\dot{\alpha}}\Big)L_{ij}\Big|_{\theta=0}\ . \tag{1.2}$$

It is invariant under the $\mathcal{N}=2$ super-Poincaré transformations, including the central charge one. The name 'linear multiplet' was coined by Breitenlohner and Sohnius [3] because the decomposition of L^{ij} into $\mathcal{N}=1$ superfields contains a real linear multiplet [4] (which is the field strength of the $\mathcal{N}=1$ tensor multiplet [5]) in the case that L^{ij} is neutral under the central charge, $\Delta L^{ij}=0$. Unlike the hypermultiplet, demanding $\Delta L^{ij}=0$ does not lead to an on-shell multiplet. The resulting off-shell multiplet without central charge [6] is naturally interpreted as the field strength of the massless $\mathcal{N}=2$ tensor multiplet [7].

The action (1.2) may be thought of as an $\mathcal{N}=2$ analogue of the chiral action in $\mathcal{N}=1$ supersymmetry. As is well known, any $\mathcal{N}=1$ action can be rewritten as a chiral one. The situation in $\mathcal{N}=2$ supersymmetry is similar. As stated by Breitenlohner and Sohnius [3], all known Lagrangians (*at that time*) for rigid $\mathcal{N}=2$ supersymmetry can be generated from linear multiplets. Since the linear multiplet was lifted to $\mathcal{N}=2$ supergravity [3], and then reformulated [8] within the $\mathcal{N}=2$ superconfomal tensor calculus [9, 10, 11], it has become a universal tool to construct the component actions for supergravity-matter systems, especially within the locally superconformal setting of [9, 10, 11].

In regard to the superspace practitioners, for a long time they had not expressed much interest in the linear multiplet, since there had appeared more powerful methods to construct off-shell supersymmetric actions using the harmonic [12, 13] and the projective [14, 15] superspace approaches which are based on the use of superspace $\mathbb{R}^{4|8}\times\mathbb{C}P^1$ pioneered by Rosly [16]. The situation changed in the mid-1990s when the so-called vector-tensor multiplet [17] was re-discovered by string theorists [18] to be

important in the context of string compactifications. This multiplet is analogous to the Fayet-Sohnius multiplet [19, 1] in the sense that it possesses an intrinsic central charge (i.e. the multiplet is on-shell if the central charge vanishes), and therefore its dynamics (including its couplings to vector multiplets and supergravity) should be described by a linear multiplet Lagrangian. The vector-tensor multiplet and its nonlinear version [20, 21] have become the subject of various studies in flat superspace [22, 23, 24, 25, 26, 27, 28]. In particular, a general harmonic superspace formalism for 4D $\mathcal{N}=2$ rigid supersymmetric theories with gauged central charge was developed in [28]. Furthermore, a remarkable construction was given by Theis [29, 30]. He proposed a new nonlinear vector-tensor multiplet with the defining properly that the central charge is gauged using the vector field belonging to the multiplet (unlike the approach of [28] which used an off-shell vector multiplet to gauge the central charge).

The action (1.2) can be represented as a superspace integral [24], but this requires, in the case $\Delta L^{ij} \neq 0$, the use of harmonic superspace [12]. Introducing SU(2) harmonics u^{+i} and u^-_i according to [12], one can associate with L^{ij} the following *analytic* superfield $L^{++} := u^+_i u^+_j L^{ij}$ which is annihilated by $D^+_\alpha := u^+_i D^i_\alpha$ and $\bar{D}^+_{\dot\alpha} := u^+_i \bar{D}^i_{\dot\alpha}$. Then, the action (1.2) is equivalent to

$$S = \int \mathrm{d}u\, \mathrm{d}\zeta^{(-4)} \Big(\theta^{+\alpha}\theta^+_\alpha + \bar\theta^+_{\dot\alpha}\bar\theta^{+\dot\alpha}\Big) L^{++} \ , \tag{1.3}$$

where $\theta^+_\alpha := u^+_i \theta^i_\alpha$ and $\bar\theta^+_{\dot\alpha} := u^+_i \bar\theta^i_{\dot\alpha}$. The integration in (1.3) is carried over the analytic subspace of the harmonic superspace. In particular, $\mathrm{d}u$ denotes the left-right invariant measure of SU(2), and

$$\mathrm{d}\zeta^{(-4)} = \frac{1}{16}\mathrm{d}^4x D^{-\alpha} D^-_\alpha \bar{D}^-_{\dot\alpha} \bar{D}^{-\dot\alpha} \ , \qquad D^-_\alpha := u^-_i D^i_\alpha \ , \quad \bar{D}^-_{\dot\alpha} := u^-_i \bar{D}^i_{\dot\alpha} \ . \tag{1.4}$$

The supergravity extension of (1.3) was given in [31]

$$S_{\rm SUGRA} = \int \mathrm{d}u\, \mathrm{d}\zeta^{(-4)}\, \mathcal{V}_5^{++} \mathcal{L}^{++} \ . \tag{1.5}$$

Here $\mathcal{L}^{++}$ denotes the curved-superspace version of the linear multiplet, while $\mathcal{V}_5^{++}$ is the harmonic prepotential of one of the two supergravity compensators – a vector multiplet which gauges the central charge. The action (1.5) is a locally supersymmetric extension of the action introduced in [28]. The combination $V_5^{++} := (\theta^+)^2 + (\bar\theta^+)^2$ in (1.3) can be interpreted as the analytic prepotential of a frozen vector multiplet with constant field strength [28]. The functional (1.5) is extremely compact and geometric, as compared with its component counterpart [8] (see eq. (3.40) below). Remarkably, $S_{\rm SUGRA}$ is a BF-type model invariant under gauge transformations of the form [31]:

$$\delta\mathcal{V}_5^{++} = -\mathcal{D}^{++}\lambda \ , \qquad \delta\mathcal{L}^{++} = \lambda\, \Delta\mathcal{L}^{++} \ , \tag{1.6}$$

with λ an analytic gauge parameter, and $\mathcal{D}^{++}$ a harmonic gauge-covariant derivative defined in [31]. Unfortunately, the above action is not yet useful for practical applications. The point is that the harmonic superspace formulation of $\mathcal{N} = 2$ supergravity was developed in terms of certain prepotentials [32, 33] (see also [13] for a review). It is not known how to derive the prepotential description of [32, 33] from the three existing superspace formulations for 4D $\mathcal{N} = 2$ conformal supergravity [34, 35, 36].[1] These formulations are realized in terms of covariant derivatives defined on a curved $\mathcal{N} = 2$ superspace. The difference between the three formulations lies in the structure groups chosen. What is important is that all known multiplets with gauged central charge in the presence of supergravity are realized in curved superspace in terms of the supergravity covariant derivatives [38, 39], and not in terms of the harmonic prepotentials. Therefore, we need a reformulation of the linear multiplet action (1.5) that is given solely in terms of the supergravity covariant derivatives. Such a reformulation is given in the present paper.

Our work contains two main results. Firstly, we develop a superform formulation for the linear multiplet in $\mathcal{N} = 2$ conformal supergravity. This formulation is shown to immediately lead to a locally supersymmetric action if we make use of the so-called ectoplasm formalism [40, 41] (also known as the superform approach to the construction of supersymmetric invariants).[2] The action derived coincides with that introduced in [8]. Secondly, we propose a new locally supersymmetric action which makes use of a deformed linear multiplet. The novel feature of this multiplet is that it corresponds to the case of the central charge being gauged using a one-form potential which is not annihilated by the central charge (unlike the standard $\mathcal{N} = 2$ vector multiplet).

This paper is organized as follows. Section 2 describes a warm-up construction. We start from a superform realization for the linear multiplet without a central charge in 5D $\mathcal{N} = 1$ Minkowski superspace, and use it to read off a superform formulation for the linear multiplet in flat 4D $\mathcal{N} = 2$ central charge superspace. In section 3 we provide a superform formulation for the linear multiplet in $\mathcal{N} = 2$ conformal super-

[1] As shown in [37], the formulation developed in [35] can be obtained from [34] by a partial gauge fixing of the super-Weyl invariance. The latter formulation is a gauged-fixed version of the conformal supergravity formulation developed in [36]. One can think of the formulation [36] as a master one. Depending on a concrete application, it is convenient to use either [35] or [36].

[2] The mathematical construction underlying the ectoplasm formalism [40, 41] is a special case of the theory of integration over surfaces in supermanifolds, see [42] and references therein. In the physics literature, the idea to use closed super four-forms for the construction of locally supersymmetric actions in 4D was, to the best of our knowledge, first given by Hasler [43] building on the analysis in [44].

gravity and derive the corresponding action functional using the ectoplasm method. In section 4 we first review, following [45], the curved-superspace formulation for a generalized $\mathcal{N}=2$ vector multiplet which gauges the central charge and is not inert under the central charge transformations (unlike the standard $\mathcal{N}=2$ vector multiplet). We then develop a superform formulation for a deformed linear multiplet and construct the associated locally supersymmetric action. The main body of the paper is accompanied by two appendices. The first appendix is technical and devoted to a brief summary of the superspace formulation for $\mathcal{N}=2$ conformal supergravity developed in [36] and slightly reformulated in [39]. The second appendix briefly describes the ectoplasm formulation of the BF coupling in $\mathcal{N}=1$ conformal supergravity.

2 The linear multiplet in flat superspace

In this section, we briefly discuss the linear multiplet L^{ij} in flat superspace and describe its superform structure. It is well known that the linear multiplet in 4D with a central charge is related to a linear multiplet in 5D without a central charge.[3] We will first describe the situation in 5D and then demonstrate its equivalence to the 4D case with a central charge.

2.1 The linear multiplet in flat 5D superspace

We use the 5D superspace and gamma matrix conventions of [46], to which we refer the reader. The algebra of 5D flat covariant derivatives[4] is

$$\{D_{\hat\alpha}^i, D_{\hat\beta}^j\} = -2\mathrm{i}\,\varepsilon^{ij}(\Gamma^{\hat c})_{\hat\alpha\hat\beta}\partial_{\hat c}\ , \qquad [D_{\hat\alpha}^i, \partial_{\hat b}] = 0\ , \qquad [\partial_{\hat a}, \partial_{\hat b}] = 0\ . \tag{2.1}$$

The linear multiplet in 5D is encoded in a real linear superfield $L^{ij} = (L_{ij})^*$ which is symmetric in its indices, $L^{ij} = L^{ji}$, and obeys the constraints

$$D_{\hat\alpha}^{(i} L^{jk)} = 0\ . \tag{2.2}$$

These constraints imply the existence of a conserved vector among the components of L^{ij},

$$V^{\hat a} = \frac{\mathrm{i}}{24}(\Gamma^{\hat a})^{\hat\alpha\hat\beta} D_{\hat\alpha j} D_{\hat\beta k} L^{jk}|\ , \qquad \partial_{\hat a} V^{\hat a} = 0\ . \tag{2.3}$$

[3] In 5D, a linear multiplet without central charge also has been called a tensor multiplet or an $\mathcal{O}(2)$ multiplet in the literature, in analogy to the terminology used in 4D.

[4] The 5D flat covariant derivatives are $D_{\hat A} = (\partial_{\hat a}, D_{\underline{\hat\alpha}})$, where $D_{\underline{\hat\alpha}} := D_{\hat\alpha}^i$. The dual basis of one-forms is $E^{\hat A} = (E^{\hat a}, E^{\underline{\hat\alpha}})$, where $E^{\underline{\hat\alpha}} := E_i^{\hat\alpha}$.

The conserved vector $V^{\hat{a}}$ is naturally dual to a closed four-form.

It is useful to introduce a superspace generalization of this four-form so that the linearity constraint (2.2) appears naturally as a Bianchi identity. Let $\widehat{\Sigma}$ be a closed four-form[5] with a tangent frame expansion

$$\widehat{\Sigma} = \frac{1}{4!} E^{\hat{D}} \wedge E^{\hat{C}} \wedge E^{\hat{B}} \wedge E^{\hat{A}} \, \widehat{\Sigma}_{\hat{A}\hat{B}\hat{C}\hat{D}} \ . \tag{2.4}$$

The requirement that $\widehat{\Sigma}$ is closed, $\mathrm{d}\widehat{\Sigma} = 0$, amounts to the equations

$$0 = D_{[\hat{A}} \widehat{\Sigma}_{\hat{B}\hat{C}\hat{D}\hat{E}\}} - 2 T_{[\hat{A}\hat{B}|}{}^{\hat{F}} \widehat{\Sigma}_{\hat{F}|\hat{C}\hat{D}\hat{E}\}} \ , \tag{2.5}$$

where the indices $\hat{A} \cdots \hat{E}$ are *graded* anti-symmetrized. Imposing the constraints

$$\widehat{\Sigma}_{\underline{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}} = 0 \ , \qquad \widehat{\Sigma}_{\hat{a}\underline{\hat{\beta}\hat{\gamma}\hat{\delta}}} = 0 \ , \qquad \widehat{\Sigma}_{\hat{a}\hat{b}}{}_{\hat{\alpha}}^{i}{}_{\hat{\beta}}^{j} = 4\mathrm{i} (\Sigma_{\hat{a}\hat{b}})_{\hat{\alpha}\hat{\beta}} L^{ij} \ , \tag{2.6}$$

for some real symmetric tensor L^{ij}, we find that the Bianchi identities require (2.2) and fix the remaining components of the four-form:

$$\widehat{\Sigma}_{\hat{a}\hat{b}\hat{c}}{}_{\hat{\alpha}}^{i} = -\frac{1}{3} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} (\Sigma^{\hat{d}\hat{e}})_{\hat{\alpha}}{}^{\hat{\beta}} D_{\hat{\beta}j} L^{ji} \ , \qquad \widehat{\Sigma}_{\hat{a}\hat{b}\hat{c}\hat{d}} = \frac{\mathrm{i}}{24} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}} (\Gamma^{\hat{e}})^{\hat{\alpha}\hat{\beta}} D_{\hat{\alpha}j} D_{\hat{\beta}k} L^{jk} \ . \tag{2.7}$$

The highest component $\widehat{\Sigma}_{\hat{a}\hat{b}\hat{c}\hat{d}}$ is closed by construction, as a consequence of the linearity constraint (2.2). This closed four-form has recently appeared in the literature [47].

It is possible to require that $\widehat{\Sigma}$ be an exact form, $\widehat{\Sigma} = \mathrm{d}\widehat{C}$, for some three-form $\widehat{C}$. In the tangent frame,

$$\widehat{\Sigma}_{\hat{A}\hat{B}\hat{C}\hat{D}} = 4 D_{[\hat{A}} \widehat{C}_{\hat{B}\hat{C}\hat{D}\}} - 6 T_{[\hat{A}\hat{B}|}{}^{\hat{E}} \widehat{C}_{\hat{E}|\hat{C}\hat{D}\}} \ . \tag{2.8}$$

Then the highest component $\widehat{\Sigma}_{\hat{a}\hat{b}\hat{c}\hat{d}}$ is similarly exact.

2.2 The linear multiplet in flat 4D central charge superspace

The 5D derivatives can be decomposed into 4D $\mathcal{N} = 2$ derivatives $D_A = (D_\alpha^i, \bar{D}_i^{\dot{\alpha}}, \partial_a)$ and a central charge ∂_5,

$$D_{\hat{\alpha}}^i = (D_\alpha^i, \bar{D}^{\dot{\alpha} i}) \ , \qquad \partial_{\hat{a}} = (\partial_a, \partial_5) \ , \tag{2.9}$$

[5]We place a hat on Σ to distinguish it from the four-form Σ in 4D that we will introduce in the next section.

so that the supersymmetry algebra becomes

$$\{D^i_\alpha, D^j_\beta\} = 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\partial_5 \ , \quad \{\bar D_{\dot\alpha i}, \bar D_{\dot\beta j}\} = 2\varepsilon_{ij}\varepsilon_{\dot\alpha\dot\beta}\partial_5 \ , \quad [D^i_\alpha, \bar D_{\dot\beta j}] = -2{\rm i}\,\delta^i{}_j(\sigma^c)_{\alpha\dot\beta}\partial_c \ ,$$
$$[D^i_\alpha, \partial_b] = [\partial_a, \partial_b] = [D^i_\alpha, \partial_5] = [\partial_a, \partial_5] = 0 \ . \tag{2.10}$$

Any multiplet in flat 5D $\mathcal{N} = 1$ superspace can naturally be written in 4D $\mathcal{N} = 2$ superspace with a real central charge $\Delta = \partial_5$. The linear multiplet L^{ij}, for example, now obeys $D^{(i}_\alpha L^{jk)} = \bar D^{(i}_{\dot\alpha} L^{jk)} = 0$. Its associated four-form multiplet $\Sigma_{\hat A\hat B\hat C\hat D}$ naturally decomposes into a four-form Σ_{ABCD} and a three-form $H_{ABC} = \Sigma_{5ABC}$, which are related by the 4D version of eq. (2.5),

$$0 = D_{[A}\Sigma_{BCDE\}} - 2T_{[AB|}{}^F\Sigma_{F|CDE\}} - 2T_{[AB|}{}^5 H_{|CDE\}} \ , \tag{2.11}$$
$$0 = \partial_5\Sigma_{ABCD} - 4D_{[A}H_{BCD\}} + 2T_{[AB|}{}^E H_{E|CD\}} \ . \tag{2.12}$$

The 5D torsion $T^5 := E^B \wedge E^A T_{AB}{}^5$ can be interpreted as the field strength of a frozen vector multiplet associated with the central charge. In form notation, these equations become

$$D\Sigma = H \wedge T^5 \ , \qquad DH = \partial_5\Sigma \ , \qquad D := E^A D_A \ , \tag{2.13}$$

where D is the central charge covariant exterior derivative, obeying $D^2 = T^5\partial_5$. The three-form H has the components

$$H_{\underline{\alpha\beta\gamma}} = H_{\underline{\alpha\beta\dot\gamma}} = 0 \ , \qquad H_a{}^i{}_\beta{}^j{}_\gamma = H_a{}^i{}_{\dot\beta}{}^j{}_{\dot\gamma} = 0 \ , \qquad H_a{}^i{}_\beta{}^j{}_{\dot\gamma} = 2(\sigma_a)_{\beta\dot\gamma}L^{ij} \ , \tag{2.14a}$$
$$H_{ab}{}^i_\alpha = -\frac{2{\rm i}}{3}(\sigma_{ab})_\alpha{}^\beta D_{\beta j}L^{ji} \ , \qquad H_{ab}{}_i^{\dot\alpha} = -\frac{2{\rm i}}{3}(\tilde\sigma_{ab})^{\dot\alpha}{}_{\dot\beta}\bar D^{\dot\beta j}L_{ji} \ , \tag{2.14b}$$
$$H_{abc} = \frac{\rm i}{24}\varepsilon_{abcd}(\tilde\sigma^d)^{\dot\beta\alpha}[D_{\alpha i}, \bar D_{\dot\beta j}]L^{ij} \ , \tag{2.14c}$$

and the four-form Σ is[6]

$$\Sigma_{\underline{\hat\alpha\hat\beta\hat\gamma\hat\delta}} = 0 \ , \qquad \Sigma_{a\underline{\hat\beta\hat\gamma\hat\delta}} = 0 \ , \qquad \Sigma_{ab}{}^i_\alpha{}^j_{\dot\beta} = 0 \ , \qquad \Sigma_{ab}{}^i_\alpha{}^j_\beta = 4{\rm i}(\sigma_{ab})_{\alpha\beta}L^{ij} \ , \tag{2.15a}$$
$$\Sigma_{abc}{}^i_\alpha = -\frac{\rm i}{3}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot\beta}\bar D^{\dot\beta}_j L^{ji} \ , \qquad \Sigma_{abcd} = \frac{1}{24}\varepsilon_{abcd}(D_{ij}L^{ij} + \bar D_{ij}L^{ij}) \ , \tag{2.15b}$$

where $D^{ij} := D^{\alpha(i}D^{j)}_\alpha$ and $\bar D^{ij} := \bar D^{(i}_{\dot\alpha}\bar D^{\dot\alpha j)}$. One can check that the closure condition $(DH)_{abcd} = \partial_5\Sigma_{abcd}$ amounts to

$$-{\rm i}\partial^{\dot\beta\alpha}[D^i_\alpha, \bar D^j_{\dot\beta}]L_{ij} = \partial_5(D^{ij}L_{ij} + \bar D^{ij}L_{ij}) \ . \tag{2.16}$$

[6]In 4D, we use a hatted spinor index to denote a four component spinor, *e.g.* $\psi_{\hat\alpha} = (\psi_\alpha, \bar\psi^{\dot\alpha})$.

What meaning can we give to these forms? The highest component of the four-form, Σ_{abcd}, has an immediate physical interpretation: it is the Sohnius Lagrangian (1.2), which associates to any linear multiplet L^{ij} a supersymmetric action principle.

The meaning of H, on the other hand, is clearest if we restrict to the case where L^{ij} is independent of the central charge, $\partial_5 L^{ij} = 0$. Then the linear multiplet becomes a tensor multiplet. In this case H is a closed three-form $DH = \mathrm{d}H = 0$, and its components (2.14) coincide with the usual encoding of a tensor multiplet into a closed three-form geometry. In particular, H_{abc} is a closed three-form and dual to a conserved vector since the right-hand side of (2.16) vanishes. In this case, H is usually interpreted as the field strength of a two-form B.

Just as in 5D, we may restrict to the case where these superforms are exact. The three-form potential $\widehat{C}$ in 5D decomposes in 4D into a three-form potential C and a two-form B,

$$C_{ABC} = \widehat{C}_{ABC} \ , \qquad B_{AB} = -\widehat{C}_{5AB} \ , \tag{2.17}$$

so that H and Σ are given respectively by

$$H_{ABC} = 3D_{[A}B_{BC\}} - 3T_{[AB|}{}^{D}B_{D|C\}} + \Delta C_{ABC} \ , \tag{2.18a}$$

$$\Sigma_{ABCD} = 4D_{[A}C_{BCD\}} - 6T_{[AB|}{}^{E}C_{E|CD\}} + 6T_{[AB}{}^{5}B_{CD\}} \ , \tag{2.18b}$$

or, equivalently,

$$H = DB + \Delta C \ , \qquad \Sigma = DC + B \wedge T^5 \ . \tag{2.19}$$

These equations automatically satisfy the Bianchi identities (2.13). An interesting consequence of the exactness condition is that one of the auxiliary components of the linear multiplet becomes the dual of a four-form field strength,

$$D_{ij}L^{ij} + \bar{D}_{ij}L^{ij} = -4\varepsilon^{abcd}\partial_a C_{bcd} \ . \tag{2.20}$$

This equation holds even in the absence of a central charge, where it describes a variant representation of the 4D $\mathcal{N} = 2$ tensor multiplet obtained from the latter by replacing one of its auxiliary scalars by the field strength of a gauge three-form [43].[7]

It turns out that the action (1.2) can be coupled to conformal supergravity. This requires that the central charge be gauged, and the usual way this is done is with an off-shell vector multiplet. The locally supersymmetric version of the action (1.2)

[7]This variant representation of $\mathcal{N} = 2$ supersymmetry was called the "three-form multiplet" in [43], by analogy with its $\mathcal{N} = 1$ counterpart constructed by Gates [48].

then corresponds to the bilinear coupling between a linear multiplet and the vector multiplet that gauges the central charge.[8] It is natural to ask how much of the above structure survives in the presence of supergravity – and the answer turns out to be all of it! In the next section, we will demonstrate how to construct a four-form Σ and three-form H in the presence of 4D conformal supergravity with a central charge and explain how the four-form Σ leads to the linear multiplet action principle [8].

3 The linear multiplet in conformal supergravity

It is well-known how to couple the linear multiplet (without a central charge) to 5D $\mathcal{N} = 1$ conformal supergravity both in superspace [49] and at the component level [50]. As our interest is mainly in its 4D manifestation, the most natural line of attack would be to construct its superform in 5D superspace and then recast 5D superspace as 4D superspace with a central charge. However, there is as yet no method to reduce 5D superspace to 4D in the presence of supergravity; indeed, this has been understood at the component level only recently [51], where it was shown explicitly that off-shell 5D conformal supergravity corresponds to off-shell 4D conformal supergravity with an additional vector multiplet. Therefore, instead of performing the reduction of the linear multiplet directly, we will begin first in four dimensions and consider the coupling of the linear multiplet to 4D conformal supergravity with a central charge.

There are several superspace formulations of 4D conformal supergravity, depending on the choice of the superspace gauge group. The formulation developed in [35] gauges $\mathrm{SO}(3,1)\times\mathrm{SU}(2)_{\mathrm{R}}$ and can be derived [37] from a formulation [34] which gauges $\mathrm{SO}(3,1) \times \mathrm{U}(2)_{\mathrm{R}}$. Neither of these explicitly gauges dilatations or special superconformal transformations; rather, both admit a super-Weyl invariance under which the various connections and torsion superfields transform in a nonlinear fashion. A more general superspace formulation exists [36] which gauges the full superconformal group (the other approaches [34] and [35] can be obtained from [36] by imposing appropriate gauge conditions, see [36] for more details).[9] This superspace formulation, which has been called $\mathcal{N} = 2$ conformal superspace, is convenient to use only when multiplets

[8] When the linear multiplet is independent of the central charge, the action is just the supersymmetric generalization of the topological BF coupling.

[9] When enlarging the structure group from $\mathrm{SU}(2)_{\mathrm{R}}$ [35] to $\mathrm{U}(2)_{\mathrm{R}}$ [34], the algebra of covariant derivatives becomes more complicated and practically unsuitable for calculations. One might think that enlarging the structure group further to the full superconformal group would make the algebra unmanageable; instead, the algebra magically simplifies [36]. This is one of the main advantages of this formulation.

and actions transform in a well-defined way under the full superconformal group. The linear multiplet falls into this class.

Throughout this paper, we make use of the superspace formulation [36] for $\mathcal{N} = 2$ conformal supergravity. All of our results derived below can be extended to the other two formulations, given in [34] and [35], by performing an appropriate gauge fixing as described in [36].

The superspace is described by a supermanifold $\mathcal{M}^{4|8}$ parametrized by local bosonic (x) coordinates and local fermionic $(\theta, \bar\theta)$ coordinates $z^M = (x^m, \theta^\mu_\imath, \bar\theta^\imath_{\dot\mu})$. The covariant derivative $\nabla_A = (\nabla_a, \nabla^i_\alpha, \bar\nabla^{\dot\alpha}_i)$ is given by

$$\nabla_A = E_A + \frac{1}{2}\Omega_A{}^{ab}M_{ab} + \Phi_A{}^{ij}J_{ij} + \mathrm{i}\Phi_A Y + B_A\mathbb{D} + \mathfrak{F}_A{}^B K_B \ , \tag{3.1}$$

with $E_A = E_A{}^M\partial_M$ the vielbein, Ω_A the spin connection, $\Phi_A{}^{ij}$ and Φ_A the $\mathrm{SU}(2)_\mathrm{R}$ and $\mathrm{U}(1)_\mathrm{R}$ connections, B_A the dilatation connection, and $\mathfrak{F}_A{}^B$ the special superconformal connection. We may extend the superspace to include a gauged central charge Δ, $[\Delta, \nabla_A] = 0$, which commutes with the superconformal generators,

$$0 = [M_{ab}, \Delta] = [J^{ij}, \Delta] = [Y, \Delta] = [\mathbb{D}, \Delta] = [K^A, \Delta] \ , \tag{3.2}$$

by introducing gauge covariant derivatives

$$\boldsymbol{\nabla}_A := \nabla_A + V_A\Delta \ , \qquad \Delta V_A = 0 \quad \longrightarrow \quad [\Delta, \boldsymbol{\nabla}_A] = 0 \ . \tag{3.3}$$

The gauge transformation of the connection V_A is

$$\delta V_A = -\nabla_A\Lambda \quad \longrightarrow \quad \delta\boldsymbol{\nabla}_A = [\Lambda\Delta, \boldsymbol{\nabla}_A] \ , \qquad \Delta\Lambda = 0 \ , \tag{3.4}$$

while the other connections, $E_A{}^M$, $\Omega_A{}^{bc}$, etc. are inert under the central charge transformation. We require a tensor superfield Ψ and its central charge descendants, $\Delta\Psi, \Delta^2\Psi, \dots$, to transform covariantly under the central charge

$$\delta_\Lambda\Psi = \Lambda\Delta\Psi \ , \qquad \delta_\Lambda\Delta\Psi = \Lambda\Delta^2\Psi \ , \quad \dots \ , \tag{3.5}$$

which implies that the central charge gauge parameter should itself be inert,

$$\delta_\Lambda\Delta = [\Lambda\Delta, \Delta] = 0 \quad \Longleftrightarrow \quad \Delta\Lambda = 0 \ . \tag{3.6}$$

Provided appropriate constraints are imposed [2], the one-form V_A describes a vector multiplet whose field strength is the reduced chiral superfield $\mathcal{Z}$. For further details and the algebra of covariant derivatives, we refer the reader to Appendix A as well as the references [36, 39].

It is possible to interpret the central charge Δ as a derivative in a fifth bosonic direction, which can simplify some of the equations we will encounter. Let $z^{\hat{M}}$ denote the coordinates of the superspace $\mathcal{M}^{4|8} \times \mathcal{X}$ where $\mathcal{M}$ is a four-dimensional $\mathcal{N} = 2$ supermanifold parametrized by coordinates z^M and $\mathcal{X}$ denotes the central charge space parametrized by x^5. The vielbein on this supermanifold is given by

$$E_{\hat{M}}{}^{\hat{A}} = \begin{pmatrix} E_M{}^A & -V_M \\ 0 & 1 \end{pmatrix} , \qquad E_{\hat{A}}{}^{\hat{M}} = \begin{pmatrix} E_A{}^M & V_A \\ 0 & 1 \end{pmatrix} , \tag{3.7}$$

and depends only on the coordinates z^M parametrizing $\mathcal{M}^{4|8}$. The connections associated with the rest of the superconformal group are completely localized on $\mathcal{M}^{4|8}$,

$$\Omega_{\hat{A}}{}^{ab} = (\Omega_A{}^{ab}, 0) , \qquad \Phi_{\hat{A}}{}^{ij} = (\Phi_A{}^{ij}, 0) , \qquad \text{etc.} , \tag{3.8a}$$

$$\partial_5 \Omega_A{}^{ab} = 0 , \qquad \partial_5 \Phi_A{}^{ij} = 0, \qquad \text{etc.} \tag{3.8b}$$

This choice for the vielbein and the other connections is preserved so long as we restrict to x^5-independent gauge transformations. We may then define

$$\widehat{\boldsymbol{\nabla}}_{\hat{A}} := E_{\hat{A}}{}^{\hat{M}} \partial_{\hat{M}} + \frac{1}{2} \Omega_{\hat{A}}{}^{ab} M_{ab} + \Phi_{\hat{A}}{}^{ij} J_{ij} + \mathrm{i} \Phi_{\hat{A}} Y + B_{\hat{A}} \mathbb{D} + \mathfrak{F}_{\hat{A}}{}^B K_B , \tag{3.9}$$

which possesses the algebra

$$\begin{aligned} [\widehat{\boldsymbol{\nabla}}_{\hat{A}}, \widehat{\boldsymbol{\nabla}}_{\hat{B}}\} = {} & T_{\hat{A}\hat{B}}{}^{\hat{C}} \widehat{\boldsymbol{\nabla}}_{\hat{C}} + \frac{1}{2} R_{\hat{A}\hat{B}}{}^{cd} M_{cd} + R_{\hat{A}\hat{B}}{}^{kl} J_{kl} \\ & + \mathrm{i} R_{\hat{A}\hat{B}}(Y) Y + R_{\hat{A}\hat{B}}(\mathbb{D}) \mathbb{D} + R_{\hat{A}\hat{B}}{}^C K_C . \end{aligned} \tag{3.10}$$

Given the choices we have made for the vielbein and the connections, it is easy to see that

$$\widehat{\boldsymbol{\nabla}}_A = \boldsymbol{\nabla}_A , \qquad \widehat{\boldsymbol{\nabla}}_5 = \partial_5 = \Delta , \tag{3.11}$$

and so the algebra of covariant derivatives (3.10) becomes

$$\begin{aligned} [\boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B\} = {} & T_{AB}{}^C \boldsymbol{\nabla}_C + F_{AB} \Delta + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^{kl} J_{kl} \\ & + \mathrm{i} R_{AB}(Y) Y + R_{AB}(\mathbb{D}) \mathbb{D} + R_{AB}{}^C K_C , \end{aligned} \tag{3.12a}$$

$$[\Delta, \boldsymbol{\nabla}_A] = 0 , \tag{3.12b}$$

provided we make the identification $F_{AB} = T_{AB}{}^5$. This leads to

$$F = \mathrm{d}V \quad \Longleftrightarrow \quad F_{AB} = 2 \boldsymbol{\nabla}_{[A} V_{B\}} - T_{AB}{}^C V_C , \tag{3.13}$$

with the Bianchi identity

$$\mathrm{d}F = 0 \quad \Longleftrightarrow \quad \boldsymbol{\nabla}_{[A}F_{BC\}} - T_{[AB|}{}^{D}F_{D|C\}} = 0 \ . \tag{3.14}$$

The algebra (3.12) is exactly that described in Appendix A. Naturally, the central charge gauge transformation arises from a diffeomorphism in the x^5-direction and must be *independent* of x^5 to preserve the form (3.7) for the vielbein. It should be kept in mind that although this superspace is formally five-dimensional, it describes only 4D $\mathcal{N} = 2$ conformal supergravity with a central charge, and *not* 5D conformal supergravity. We will refer to this superspace as central charge superspace.[10]

We generalize the flat linear multiplet by introducing a closed superspace four-form $\widehat{\Sigma}$ in central charge superspace,

$$\hat{\mathrm{d}}\widehat{\Sigma} = 0 \quad \Longleftrightarrow \quad \widehat{\boldsymbol{\nabla}}_{[\hat{A}}\widehat{\Sigma}_{\hat{B}\hat{C}\hat{D}\hat{E}\}} - 2T_{[\hat{A}\hat{B}|}{}^{\hat{F}}\widehat{\Sigma}_{\hat{F}|\hat{C}\hat{D}\hat{E}\}} = 0 \ . \tag{3.15}$$

This closed form decomposes into a four-form and a three-form when written in 4D superspace. Denoting

$$\Sigma_{ABCD} = \widehat{\Sigma}_{ABCD} \ , \qquad H_{ABC} = \widehat{\Sigma}_{5ABC} \ , \tag{3.16}$$

for a four-form Σ and a three-form H, the equation (3.15) decomposes into two equations,

$$\boldsymbol{\nabla}_{[A}\Sigma_{BCDE\}} - 2T_{[AB|}{}^{F}\Sigma_{F|CDE\}} = 2F_{[AB}H_{CDE\}} \ , \tag{3.17a}$$

$$4\boldsymbol{\nabla}_{[A}H_{BCD]} - 2T_{[AB|}{}^{E}H_{E|CD\}} = \Delta\Sigma_{ABCD} \ , \tag{3.17b}$$

which may equivalently be written

$$\boldsymbol{\nabla}\Sigma = H \wedge F \ , \qquad \boldsymbol{\nabla}H = \Delta\Sigma \ , \tag{3.18}$$

where $\boldsymbol{\nabla} := E^A\boldsymbol{\nabla}_A$ is the covariant exterior derivative of 4D superspace. The superforms H and Σ are required to transform as scalars under central charge gauge transformations,

$$\delta_\Lambda H_{ABC} = \Lambda\Delta H_{ABC} \ , \qquad \delta_\Lambda \Sigma_{ABCD} = \Lambda\Delta\Sigma_{ABCD} \ . \tag{3.19}$$

Imposing the constraints

$$H_{\underline{\alpha\beta\gamma}} = H_{\underline{\alpha\beta\dot{\gamma}}} = 0 \ , \qquad H_{a}{}_{\beta}^{i}{}_{\gamma}^{j} = H_{a}{}_{\dot{\beta}}^{i}{}_{\dot{\gamma}}^{j} = 0 \ , \qquad H_{a}{}_{\beta}^{i}{}_{\dot{\gamma}}^{j} = 2(\sigma_a)_{\beta\dot{\gamma}}\mathcal{L}^{ij} \ , \tag{3.20}$$

$$\Sigma_{\underline{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}} = 0 \ , \qquad \Sigma_{a\underline{\hat{\beta}\hat{\gamma}\hat{\delta}}} = 0 \ , \qquad \Sigma_{ab}{}_{\alpha}^{i}{}_{\dot{\beta}}^{j} = 0 \ , \tag{3.21}$$

[10] A related superspace involving a complex central charge was constructed in [52].

we find that the superfield $\mathcal{L}^{ij}$ must be a linear multiplet,

$$\boldsymbol{\nabla}_\alpha^{(i}\mathcal{L}^{jk)} = \bar{\boldsymbol{\nabla}}_{\dot\alpha}^{(i}\mathcal{L}^{jk)} = 0 \ . \tag{3.22}$$

The remaining components of H are given by

$$H_{ab}{}_\alpha^i = -\frac{2\mathrm{i}}{3}(\sigma_{ab})_\alpha{}^\beta \boldsymbol{\nabla}_{\beta j}\mathcal{L}^{ji} \ , \qquad H_{abc} = \frac{\mathrm{i}}{24}\varepsilon_{abcd}(\tilde\sigma^d)^{\dot\alpha\alpha}[\boldsymbol{\nabla}_\alpha^k, \bar{\boldsymbol{\nabla}}_{\dot\alpha}^l]\mathcal{L}_{kl} \ , \tag{3.23}$$

and those of Σ are given by

$$\Sigma_{ab}{}_{\alpha\beta}^{i\,j} = 4\mathrm{i}(\sigma_{ab})_{\alpha\beta}\bar{\mathcal{Z}}\mathcal{L}^{ij} \ , \tag{3.24a}$$

$$\Sigma_{abc}{}_\alpha^i = -\frac{\mathrm{i}}{2}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot\alpha}\bar{\boldsymbol{\nabla}}_j^{\dot\alpha}\bar{\mathcal{Z}}\mathcal{L}^{ij} - \frac{\mathrm{i}}{3}\varepsilon_{abcd}(\sigma^d)_{\alpha\dot\alpha}\bar{\mathcal{Z}}\bar{\boldsymbol{\nabla}}_j^{\dot\alpha}\mathcal{L}^{ij} \ , \tag{3.24b}$$

$$\begin{aligned}\Sigma_{abcd} = \frac{1}{24}\varepsilon_{abcd}(&\mathcal{Z}\boldsymbol{\nabla}_{kl}\mathcal{L}^{kl} + \bar{\mathcal{Z}}\bar{\boldsymbol{\nabla}}_{kl}\mathcal{L}^{kl} + 3\boldsymbol{\nabla}^{kl}\mathcal{Z}\mathcal{L}_{kl} \\ &+ 4\boldsymbol{\nabla}^{\gamma k}\mathcal{Z}\boldsymbol{\nabla}_\gamma^l\mathcal{L}_{kl} + 4\bar{\boldsymbol{\nabla}}_{\dot\gamma k}\bar{\mathcal{Z}}\bar{\boldsymbol{\nabla}}_l^{\dot\gamma}\mathcal{L}^{kl}) \ .\end{aligned} \tag{3.24c}$$

These results can be compared with those in the previous section by setting $\boldsymbol{\nabla}_A \to D_A$ and $\mathcal{Z} \to 1$.

As in the flat case, two special situations are noteworthy. The first is if $\mathcal{L}^{ij}$ is taken to be independent of the central charge, $\Delta\mathcal{L}^{ij} = 0$, then H is closed in the usual sense,

$$\Delta\mathcal{L}^{ij} = 0 \quad \longrightarrow \quad \mathrm{d}H = 0 \ , \tag{3.25}$$

and $\mathcal{L}^{ij}$ becomes a tensor multiplet. The second situation is if we choose the closed form $\widehat\Sigma$ to be exact,

$$\widehat\Sigma = \hat{\mathrm{d}}\widehat{C} \quad \Longleftrightarrow \quad \widehat\Sigma_{\hat A\hat B\hat C\hat D} = 4\widehat{\boldsymbol{\nabla}}_{[\hat A}\widehat{C}_{\hat B\hat C\hat D\}} - 6T_{[\hat A\hat B|}{}^{\hat E}\widehat{C}_{\hat E|\hat C\hat D\}} \ . \tag{3.26}$$

This implies that H and Σ are given in terms of a two-form $B_{AB} = -\widehat{C}_{5AB}$ and a three-form $C_{ABC} = \widehat{C}_{ABC}$,

$$H_{ABC} = 3\boldsymbol{\nabla}_{[A}B_{BC\}} - 3T_{[AB|}{}^D B_{D|C\}} + \Delta C_{ABC} \ , \tag{3.27a}$$

$$\Sigma_{ABCD} = 4\boldsymbol{\nabla}_{[A}C_{BCD\}} - 6T_{[AB|}{}^E C_{E|CD\}} + 6F_{[AB}B_{CD\}} \ , \tag{3.27b}$$

or, equivalently,

$$H = \boldsymbol{\nabla}B + \Delta C \ , \qquad \Sigma = \boldsymbol{\nabla}C + B\wedge F \ . \tag{3.28}$$

As in the flat case, this leads to a variant representation for the linear multiplet in 4D where one of its auxiliaries is the divergence of a vector. The supergravity

generalization of eq. (2.20), however, is quite complicated, so we will not construct it explicitly here. In the case that the conditions (3.25) and (3.26) are imposed simultaneously, we obtain a variant realization of the tensor multiplet such that one of its auxiliaries is replaced by the field strength of a gauge three-form. We can think of this realization as a three-form multiplet in $\mathcal{N} = 2$ conformal supergravity.

Now we would like to interpret Σ_{abcd} as (part of) a supersymmetric Lagrangian. This turns out to be possible using the so-called ectoplasm formalism [40, 41]. The key element of this approach is a superspace four-form J which is closed.[11] The action constructed by integrating J over the manifold $\mathcal{M}$ parametrized by the physical coordinates x^m turns out to be automaticaly supersymmetric, which we will demonstrate shortly.

In our case, Σ is not itself closed, but we may easily construct a related four-form that is:

$$J := \Sigma + V \wedge H \ . \tag{3.29}$$

It is straightforward to check that J is closed,

$$\mathrm{d}J = \mathrm{d}\Sigma + V \wedge \mathrm{d}H - \mathrm{d}V \wedge H = \boldsymbol{\nabla}\Sigma - V \wedge \Delta\Sigma + V \wedge \boldsymbol{\nabla}H - F \wedge H = 0 \ , \tag{3.30}$$

using eqs. (3.18). We can construct a supersymmetric action via the integration of J over the manifold $\mathcal{M}$:[12]

$$S = \int_{\mathcal{M}} J = \int \mathrm{d}^4x \, e \, ({}^*J) \ , \qquad {}^*J = \frac{1}{4!}\varepsilon^{mnpq} J_{mnpq} \ . \tag{3.31}$$

This action is automatically supersymmetric by virtue of the closure of J. The proof is straightforward. Since supersymmetry is the combination of a superdiffeomorphism and a gauge transformation, it suffices to show that the action is invariant separately under superdiffeomorphisms and gauge transformations. First, we observe that a superdiffeomorphism is a super Lie derivative:

$$\delta_\xi J \equiv \mathcal{L}_\xi J \equiv \imath_\xi \mathrm{d}J + \mathrm{d}\imath_\xi J = \mathrm{d}\imath_\xi J \ , \tag{3.32}$$

with the last equality following since J is closed. Provided that the manifold $\mathcal{M}$ has no boundary, the variation of the action is zero. Next, we consider gauge transformations. Since J is a scalar under the superconformal generators (Lorentz, $\mathrm{U}(2)_\mathrm{R}$, dilatation

[11] On a usual four-dimensional manifold, any four-form is closed trivially, but in superspace the condition is nontrivial.

[12] We define the Levi-Civita tensor with world indices as $\varepsilon^{mnpq} := \varepsilon^{abcd} e_a{}^m e_b{}^n e_c{}^p e_d{}^q$.

and special superconformal), the only nontrivial check involves the central charge gauge transformation. We note that

$$\delta_\Lambda J = \delta_\Lambda \Sigma + \delta_\Lambda V \wedge H + V \wedge \delta_\Lambda H \ , \tag{3.33}$$

but Σ and H both transform covariantly under central charge gauge transformations, $\delta_\Lambda \Sigma = \Lambda \Delta \Sigma$ and $\delta_\Lambda H = \Lambda \Delta H$, while V transforms as a connection, $\delta_\Lambda V = -\mathrm{d}\Lambda$. So we find

$$\delta_\Lambda J = \Lambda \Delta \Sigma - \mathrm{d}\Lambda \wedge H + V \wedge \Lambda \Delta H = \mathrm{d}(\Lambda H) \ . \tag{3.34}$$

Once again J transforms into an exact form and so the action S is invariant.

We can now give the supersymmetric action explicitly. We identify

$$J_{mnpq} = \Sigma_{mnpq} - 4 V_{[m} H_{npq]} \tag{3.35}$$

or equivalently,

$${}^*J = \frac{1}{4!}\varepsilon^{mnpq}\Sigma_{mnpq} - \frac{1}{3!}\varepsilon^{mnpq} V_m H_{npq} \ . \tag{3.36}$$

The second term is a topological BF coupling; the first term is its supersymmetric completion and is given by

$$\begin{aligned}
\frac{1}{4!}\varepsilon^{mnpq}\Sigma_{mnpq}| &= \frac{1}{4!}\varepsilon^{mnpq} E_q{}^D E_p{}^C E_n{}^B E_m{}^A \Sigma_{ABCD}| \\
&= \frac{1}{4!}\varepsilon^{abcd}\Big(\frac{1}{2}\Sigma_{abcd} + 2\psi_{a}{}_i^{\alpha}\Sigma^i_{\alpha bcd} + \frac{3}{2}\psi_{b}{}_j^{\beta}\psi_{a}{}_i^{\alpha}\Sigma^{i\,j}_{\alpha\beta cd}\Big) + \text{c.c.} \\
&= -\frac{1}{2}F\phi - \frac{1}{2}\chi_i^\alpha\lambda_\alpha^i - \frac{1}{16}\ell^{ij}X_{ij} + \frac{\mathrm{i}}{4}\psi_{\alpha\dot\alpha}{}_i^{\alpha}(2\bar\chi^{\dot\alpha i}\bar\phi + \ell^{ij}\bar\lambda_j^{\dot\alpha}) \\
&\quad - \frac{1}{2}(\sigma^{cd})_{\gamma\delta}\psi_{c}{}_k^{\gamma}\psi_{d}{}_l^{\delta}\ell^{kl}\bar\phi + \text{c.c.} \ ,
\end{aligned} \tag{3.37}$$

where

$$\ell^{ij} := \mathcal{L}^{ij}| \ , \tag{3.38a}$$

$$\chi_{\alpha i} := \frac{1}{3}\boldsymbol{\nabla}_\alpha^j \mathcal{L}_{ij}| \ , \qquad \bar\chi^{\dot\alpha i} := \frac{1}{3}\bar{\boldsymbol{\nabla}}^{\dot\alpha}_j \mathcal{L}^{ij}| \ , \tag{3.38b}$$

$$F := \frac{1}{12}\boldsymbol{\nabla}^{ij}\mathcal{L}_{ij}| \ , \qquad \bar F := \frac{1}{12}\bar{\boldsymbol{\nabla}}^{ij}\mathcal{L}_{ij}| \ , \tag{3.38c}$$

and

$$\phi := \mathcal{Z}| \ , \quad \lambda_\alpha^i := \boldsymbol{\nabla}_\alpha^i \mathcal{Z}| \ , \quad X^{ij} := \boldsymbol{\nabla}^{ij}\mathcal{Z}| \ . \tag{3.39}$$

The full action is

$$S = -\frac{1}{2}\int \mathrm{d}^4x\, e\, \Big(F\phi + \chi_i^\alpha \lambda_\alpha^i + \frac{1}{8}\ell^{ij} X_{ij} + \frac{1}{6}\varepsilon^{mnpq} V_m H_{npq} \\ - \frac{\mathrm{i}}{2}\psi_{\alpha\dot\alpha i}{}^\alpha (2\bar\chi^{\dot\alpha i}\bar\phi + \ell^{ij}\bar\lambda_j^{\dot\alpha}) + (\sigma^{cd})_{\gamma\delta}\psi_{ck}{}^\gamma \psi_{dl}{}^\delta \ell^{kl}\bar\phi + \text{c.c.}\Big) , \qquad (3.40)$$

which agrees with [8]. (This action is equivalent to that given in [3] up to a gauge-fixing.) The terms V_m and H_{npq} are understood as the projections of the corresponding superforms. Up to a normalization factor, this is exactly the supersymmetric action coupling a linear multiplet to the vector multiplet gauging the central charge.

4 A deformed linear multiplet

Now we turn to the main point of our paper: the generalization of the linear multiplet when the central charge is gauged by a more elaborate multiplet. We describe first a superspace where the central charge connection itself transforms under the central charge, reviewing the construction given recently in [45]. Then we reexamine the structure of the coupled four-form Σ and three-form H to discover a generalized version of the linear multiplet. This naturally implies a generalized version of the action principle (3.40).

4.1 A large vector multiplet

Until now we have gauged the central charge using a normal $\mathcal{N} = 2$ vector multiplet – that is, the vector multiplet was inert under the central charge. A generalization immediately presents itself: we may choose the central charge gauge connection to no longer be inert under Δ. We identify

$$\boldsymbol{\nabla}_A := \nabla_A + \mathcal{V}_A\Delta , \qquad \Delta\mathcal{V}_A \neq 0 , \qquad (4.1)$$

where ∇_A is the original covariant derivative of conformal supergravity, while $\mathcal{V}_A$ is the gauge connection associated with Δ. The gauge transformation of $\mathcal{V}_A$ is

$$\delta\mathcal{V}_A = -\nabla_A\Lambda + \Lambda\Delta\mathcal{V}_A \quad \longrightarrow \quad \delta\boldsymbol{\nabla}_A = [\Lambda\Delta, \boldsymbol{\nabla}_A] , \qquad \Delta\Lambda = 0 . \qquad (4.2)$$

Unlike the gauge one-form $\mathcal{V}_A$, the gauge parameter is neutral with respect to the central charge. As before, the central charge commutes with the other generators, (3.2), but because $\Delta\mathcal{V}_A \neq 0$, we find that $[\Delta, \boldsymbol{\nabla}_A] \neq 0$.

A five-dimensional interpretation is even more useful now than before. Again, we take the vielbein of the larger superspace to be

$$E_{\hat{M}}{}^{\hat{A}} = \begin{pmatrix} E_M{}^A & -\mathcal{V}_M \\ 0 & 1 \end{pmatrix} , \qquad E_{\hat{A}}{}^{\hat{M}} = \begin{pmatrix} E_A{}^M & \mathcal{V}_A \\ 0 & 1 \end{pmatrix} . \tag{4.3}$$

We allow $\mathcal{V}_A = E_A{}^M \mathcal{V}_M$ to depend on the fifth bosonic coordinate, but we take $E_A{}^M$ to be independent of x^5 as before. The connections are given again by (3.8) and the covariant derivative by (3.9), leading to

$$\widehat{\boldsymbol{\nabla}}_{\hat{A}} = (\boldsymbol{\nabla}_A, \Delta) , \qquad \Delta = \partial_5 . \tag{4.4}$$

The algebra of covariant derivatives (3.10) now decomposes into

$$\begin{aligned}[][\boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B\} &= T_{AB}{}^C \boldsymbol{\nabla}_C + \mathcal{F}_{AB}\Delta + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^{kl} J_{kl} \\ &\quad + \mathrm{i} R_{AB}(Y) Y + R_{AB}(\mathbb{D})\mathbb{D} + R_{AB}{}^C K_C , \end{aligned} \tag{4.5a}$$

$$[\Delta, \boldsymbol{\nabla}_A] = \mathcal{F}_{5A}\Delta , \tag{4.5b}$$

provided we make the identifications

$$\mathcal{F} = T^5 \qquad \Longleftrightarrow \qquad \mathcal{F}_{AB} = T_{AB}{}^5 , \quad \mathcal{F}_{5A} = T_{5A}{}^5 , \tag{4.6}$$

which leads to

$$\mathcal{F}_{AB} = 2\boldsymbol{\nabla}_{[A}\mathcal{V}_{B\}} - T_{AB}{}^C \mathcal{V}_C , \qquad \mathcal{F}_{5A} = \Delta \mathcal{V}_A = \partial_5 \mathcal{V}_A . \tag{4.7}$$

The torsion tensor T^5 is closed by construction,

$$\widehat{\boldsymbol{\nabla}} T^5 = 0 \quad \Longleftrightarrow \quad \widehat{\boldsymbol{\nabla}}_{[\hat{C}} T_{\hat{B}\hat{A}\}}{}^5 - T_{[\hat{C}\hat{B}|}{}^{\hat{D}} T_{\hat{D}|\hat{A}\}}{}^5 = 0 , \tag{4.8}$$

where $\widehat{\boldsymbol{\nabla}} := E^{\hat{A}} \widehat{\boldsymbol{\nabla}}_{\hat{A}}$. This implies similar relations for $\mathcal{F}$,

$$\boldsymbol{\nabla}_{[A}\mathcal{F}_{BC\}} - T_{[AB|}{}^D \mathcal{F}_{D|C\}} = \mathcal{F}_{[AB|}\mathcal{F}_{5|C\}} , \tag{4.9}$$

$$2\boldsymbol{\nabla}_{[A|}\mathcal{F}_{5|B\}} - T_{AB}{}^D \mathcal{F}_{5D} = \Delta \mathcal{F}_{AB} . \tag{4.10}$$

If we introduce the two-form $\mathcal{F}$ and the one-forms $\mathcal{V}$ and $\mathcal{F}_5$, defined by

$$\mathcal{F} = \frac{1}{2} E^B \wedge E^A \mathcal{F}_{AB} , \qquad \mathcal{F}_5 = E^A \mathcal{F}_{5A} , \qquad \mathcal{V} = E^A \mathcal{V}_A , \tag{4.11}$$

then $\mathcal{F}$ and $\mathcal{F}_5$ can be written

$$\mathcal{F} = \mathrm{d}\mathcal{V} + \Delta\mathcal{V} \wedge \mathcal{V} , \qquad \mathcal{F}_5 = \Delta\mathcal{V} , \tag{4.12}$$

and the Bianchi identities become

$$\boldsymbol{\nabla}\mathcal{F} \equiv \mathrm{d}\mathcal{F} + \Delta\mathcal{F}\wedge\mathcal{V} = \mathcal{F}_5\wedge\mathcal{F} \ , \qquad \boldsymbol{\nabla}\mathcal{F}_5 \equiv \mathrm{d}\mathcal{F}_5 + \Delta\mathcal{F}_5\wedge\mathcal{V} = \Delta\mathcal{F} \ , \tag{4.13}$$

with $\boldsymbol{\nabla} := E^A\boldsymbol{\nabla}_A$.

Let us now impose constraints on the field strength $\mathcal{F}$. In analogy to the x^5-independent case, we take[13]

$$\mathcal{F}_{\alpha\beta}^{i\,j} = 2\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{M} \ , \qquad \mathcal{F}^{\dot\alpha\dot\beta}_{i\ j} = -2\varepsilon^{\dot\alpha\dot\beta}\varepsilon_{ij}M \ , \qquad \mathcal{F}_{\alpha j}^{i\,\dot\beta} = 0 \ , \tag{4.14}$$

where M is a conformally primary superfield of dimension 1 and $U(1)$ charge -2. Analyzing the Bianchi identities, we find that M must obey two constraints,

$$\boldsymbol{\nabla}_\alpha^{(i}\bar{\boldsymbol{\nabla}}_{\dot\alpha}^{j)} \ln\left(\frac{M}{\bar{M}}\right) = 0 \ , \tag{4.15}$$

$$\bar{M}\boldsymbol{\nabla}^{ij}\left(\frac{M}{\bar{M}}\right) = M\bar{\boldsymbol{\nabla}}^{ij}\left(\frac{\bar{M}}{M}\right) \ . \tag{4.16}$$

The remaining components of $\mathcal{F}$ are then determined to be

$$\mathcal{F}_{a\beta}^{\ j} = -\frac{\mathrm{i}}{2}(\sigma_a)_\beta{}^{\dot\alpha}\bar{M}\bar{\boldsymbol{\nabla}}^j_{\dot\alpha}\ln\left(\frac{\bar{M}}{M}\right) \ , \tag{4.17a}$$

$$\mathcal{F}_{ab} = \frac{1}{8}(\sigma_{ab})^{\alpha\beta}(\bar{M}\boldsymbol{\nabla}_{\alpha\beta}\Big(\frac{M}{\bar{M}}\Big) + 4\bar{M}W_{\alpha\beta}) + \text{c.c.} \ , \tag{4.17b}$$

$$\mathcal{F}_5{}_\alpha^i = \boldsymbol{\nabla}_\alpha^i \ln\bar{M} \ , \tag{4.17c}$$

$$\mathcal{F}_{5a} = -\frac{\mathrm{i}}{8}(\sigma_a)_{\alpha\dot\alpha}(\boldsymbol{\nabla}^{\alpha k}\bar{\boldsymbol{\nabla}}^{\dot\alpha}_k\ln M + \bar{\boldsymbol{\nabla}}^{\dot\alpha}_k\boldsymbol{\nabla}^{\alpha k}\ln\bar{M}) \ . \tag{4.17d}$$

It is straightforward to check that if $\mathcal{V}_A$ is x^5-independent, then M becomes a reduced chiral superfield $\mathcal{Z}$.

This large vector multiplet has an interesting feature. Although $[\Delta, \boldsymbol{\nabla}_A] = \mathcal{F}_{5A}\Delta$ is nonzero, we can easily see that

$$[\bar{M}\Delta, \boldsymbol{\nabla}_\alpha^i] = 0 \ , \qquad [M\Delta, \bar{\boldsymbol{\nabla}}^{\dot\alpha}_i] = 0 \ . \tag{4.18}$$

4.2 Deformed linear multiplet

Now let us construct a deformation of the linear multiplet in four dimensions. The constraints we will impose are quite cumbersome if we insist on a purely four

[13] Our definition of M differs by a factor of i from [45].

dimensional superspace interpretation. In 4D superspace, we take a four-form Σ and a three-form H to obey the constraints

$$\boldsymbol{\nabla}\Sigma = H \wedge \mathcal{F} , \qquad \boldsymbol{\nabla} H + H \wedge \mathcal{F}_5 = \Delta\Sigma , \tag{4.19}$$

which can equivalently be written

$$\boldsymbol{\nabla}_{[A}\Sigma_{BCDE\}} - 2T_{[AB|}{}^{F}\Sigma_{F|CDE\}} = 2\mathcal{F}_{[AB}H_{CDE\}} , \tag{4.20a}$$

$$4\boldsymbol{\nabla}_{[A}H_{BCD]} - 2T_{[AB|}{}^{E}H_{E|CD\}} + 4\mathcal{F}_{5[A}H_{BCD\}} = \Delta\Sigma_{ABCD} . \tag{4.20b}$$

Central charge superspace offers a more economical way of encoding the above equations. The superforms Σ and H may be placed within a single superform $\widehat{\Sigma}$,

$$\Sigma_{ABCD} = \widehat{\Sigma}_{ABCD} , \qquad H_{ABC} = \widehat{\Sigma}_{5ABC} . \tag{4.21}$$

We require $\widehat{\Sigma}$ to be closed, which amounts to

$$0 = \widehat{\boldsymbol{\nabla}}_{[\hat{A}}\widehat{\Sigma}_{\hat{B}\hat{C}\hat{D}\hat{E}\}} - 2T_{[\hat{A}\hat{B}|}{}^{\hat{F}}\widehat{\Sigma}_{\hat{F}|\hat{C}\hat{D}\hat{E}\}} . \tag{4.22}$$

This equation is equivalent to the two equations (4.20).

By fixing some of the lowest components of Σ and H, one can show that they are completely specified by a deformed linear multiplet $\mathcal{L}^{ij}$, obeying

$$\boldsymbol{\nabla}_{\alpha}^{(i}(\bar{M}\mathcal{L}^{jk)}) = 0 , \qquad \bar{\boldsymbol{\nabla}}_{\dot{\alpha}}^{(i}(M\mathcal{L}^{jk)}) = 0 . \tag{4.23}$$

It is useful to introduce tilded derivatives defined as

$$\widetilde{\boldsymbol{\nabla}}_{\alpha}^{i} = \bar{M}^{-1}\boldsymbol{\nabla}_{\alpha}^{i}\bar{M} , \qquad \widetilde{\bar{\boldsymbol{\nabla}}}{}^{\dot{\alpha}}_{i} = M^{-1}\bar{\boldsymbol{\nabla}}{}^{\dot{\alpha}}_{i}M , \tag{4.24}$$

so the conditions (4.23) can be more compactly written

$$\widetilde{\boldsymbol{\nabla}}_{\alpha}^{(i}\mathcal{L}^{jk)} = 0 , \qquad \widetilde{\bar{\boldsymbol{\nabla}}}_{\dot{\alpha}}^{(i}\mathcal{L}^{jk)} = 0 . \tag{4.25}$$

The components of the three-form $H_{ABC} = \widehat{\Sigma}_{5ABC}$ are

$$H_{\underline{\hat{\alpha}\hat{\beta}\hat{\gamma}}} = 0 , \qquad H_{a}{}_{\beta}^{i}{}_{\gamma}^{j} = H_{a}{}_{i}^{\dot{\beta}}{}_{j}^{\dot{\gamma}} = 0 , \qquad H_{a}{}_{\beta}^{i}{}_{j}^{\dot{\gamma}} = 2(\sigma_a)_{\beta}{}^{\dot{\gamma}}\mathcal{L}^{i}{}_{j} , \tag{4.26a}$$

$$H_{ab}{}_{\alpha}^{i} = \frac{2\mathrm{i}}{3}(\sigma_{ab})_{\alpha}{}^{\gamma}\widetilde{\boldsymbol{\nabla}}_{\gamma}^{k}\mathcal{L}_{k}{}^{i} , \quad H_{ab}{}_{i}^{\dot{\alpha}} = \frac{2\mathrm{i}}{3}(\tilde{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\gamma}}\widetilde{\bar{\boldsymbol{\nabla}}}{}^{\dot{\gamma}}_{k}\mathcal{L}^{k}{}_{i} , \tag{4.26b}$$

$$H_{abc} = \frac{\mathrm{i}}{24}\varepsilon_{abcd}(\tilde{\sigma}^{d})^{\dot{\gamma}\gamma}[\widetilde{\boldsymbol{\nabla}}_{\gamma j}, \widetilde{\bar{\boldsymbol{\nabla}}}_{\dot{\gamma}k}]\mathcal{L}^{jk} \tag{4.26c}$$

and the components of the four-form $\Sigma_{ABCD} = \widehat{\Sigma}_{ABCD}$ are

$$\Sigma_{\underline{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}} = \Sigma_{a\underline{\hat{\beta}\hat{\gamma}\hat{\delta}}} = \Sigma_{ab}{}_{\alpha}^{i}{}_{j}^{\dot{\beta}} = 0 \ , \tag{4.27a}$$

$$\Sigma_{ab}{}_{\alpha}^{i}{}_{\beta}^{j} = 4\mathrm{i}(\sigma_{ab})_{\alpha\beta}\bar{M}\mathcal{L}^{ij} \ , \quad \Sigma_{ab}{}_{i}^{\dot{\alpha}}{}_{j}^{\dot{\beta}} = -4\mathrm{i}(\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}M\mathcal{L}_{ij} \ , \tag{4.27b}$$

$$\Sigma_{abc}{}_{\alpha}^{i} = -\mathrm{i}\,\varepsilon_{abcd}(\sigma^{d})_{\alpha\dot{\gamma}}\left(\frac{1}{2}M\bar{\boldsymbol{\nabla}}^{\dot{\gamma}}_{k}\Big(\frac{\bar{M}}{M}\Big)\mathcal{L}^{ki} + \frac{1}{3}\bar{M}\tilde{\bar{\boldsymbol{\nabla}}}^{\dot{\gamma}}_{k}\mathcal{L}^{ki}\right) \ , \tag{4.27c}$$

$$\Sigma_{abc}{}_{i}^{\dot{\alpha}} = -\mathrm{i}\,\varepsilon_{abcd}(\tilde{\sigma}^{d})^{\dot{\alpha}\gamma}\left(\frac{1}{2}\bar{M}\boldsymbol{\nabla}^{k}_{\gamma}\Big(\frac{M}{\bar{M}}\Big)\mathcal{L}_{ki} + \frac{1}{3}M\tilde{\boldsymbol{\nabla}}^{k}_{\gamma}\mathcal{L}_{ki}\right) \ , \tag{4.27d}$$

$$\Sigma_{abcd} = \frac{1}{24}\varepsilon_{abcd}\left(M\tilde{\boldsymbol{\nabla}}_{jk}\mathcal{L}^{jk} + 4\bar{M}\boldsymbol{\nabla}^{\gamma}_{j}\Big(\frac{M}{\bar{M}}\Big)\tilde{\boldsymbol{\nabla}}_{\gamma k}\mathcal{L}^{jk} + \frac{3}{2}\bar{M}\boldsymbol{\nabla}_{jk}\Big(\frac{M}{\bar{M}}\Big)\mathcal{L}^{jk} + \text{c.c.}\right) \ . \tag{4.27e}$$

4.3 Locally supersymmetric action

Let us apply the ectoplasm method to the four-form constructed in the previous subsection. The superform Σ obeys the equation

$$\mathrm{d}\Sigma = H \wedge \mathcal{F} - \Delta\Sigma \wedge \mathcal{V} = \mathrm{d}(H \wedge \mathcal{V}) \ , \tag{4.28}$$

so we can introduce the closed four-form

$$J := \Sigma - H \wedge \mathcal{V} \ , \tag{4.29}$$

which transforms under a central charge gauge transformation as an exact form,

$$\delta J = \Lambda\Delta\Sigma - \Lambda\Delta H \wedge \mathcal{V} - H \wedge \mathrm{d}\Lambda = \mathrm{d}(\Lambda H) \ . \tag{4.30}$$

Then by the argument made in the previous section, we may define an action using the Lagrangian

$$^{*}J := \frac{1}{4!}\varepsilon^{mnpq}J_{mnpq} = \frac{1}{4!}\varepsilon^{mnpq}\Sigma_{mnpq} - \frac{1}{3!}\varepsilon^{mnpq}\mathcal{V}_{m}H_{npq} \ . \tag{4.31}$$

This is naturally supersymmetric and gauge-invariant. The explicit action is easy to construct once we note the similarities between eqs. (3.20), (3.23) and eqs. (4.26) and also between eqs. (3.21), (3.24) and eqs. (4.27). We need only make the identifications

$$\ell^{ij} := \mathcal{L}^{ij}| \ , \tag{4.32a}$$

$$\chi_{\alpha i} := \frac{1}{3}\tilde{\boldsymbol{\nabla}}^{j}_{\alpha}\mathcal{L}_{ij}| \ , \quad \bar{\chi}^{\dot{\alpha}i} := \frac{1}{3}\tilde{\bar{\boldsymbol{\nabla}}}^{\dot{\alpha}}_{j}\mathcal{L}^{ij}| \ , \tag{4.32b}$$

$$F := \frac{1}{12}\tilde{\boldsymbol{\nabla}}^{ij}\mathcal{L}_{ij}| \ , \quad \bar{F} := \frac{1}{12}\tilde{\bar{\boldsymbol{\nabla}}}^{ij}\mathcal{L}_{ij}| \ , \tag{4.32c}$$

and

$$\phi := M| \ , \quad \lambda_\alpha^i := \bar{M}\boldsymbol{\nabla}_\alpha^i\Big(\frac{M}{\bar{M}}\Big)| \ , \quad X^{ij} := \bar{M}\boldsymbol{\nabla}^{ij}\Big(\frac{M}{\bar{M}}\Big)| \ . \tag{4.33}$$

The full action is then formally identical to the action (3.40)

$$\begin{aligned} S = -\frac{1}{2}\int \mathrm{d}^4x\, e\,\Big(&F\phi + \chi_i^\alpha\lambda_\alpha^i + \frac{1}{8}\ell^{ij}X_{ij} + \frac{1}{6}\varepsilon^{mnpq}\mathcal{V}_m H_{npq} \\ &- \frac{\mathrm{i}}{2}\psi_{\alpha\dot\alpha}{}_i^\alpha(2\bar\chi^{\dot\alpha i}\bar\phi + \ell^{ij}\bar\lambda_j^{\dot\alpha}) + (\sigma^{cd})_{\gamma\delta}\psi_c{}_k^\gamma\psi_d{}_l^\delta\ell^{kl}\bar\phi + \text{c.c.}\Big) \ . \end{aligned} \tag{4.34}$$

The difference at the component level is that the supersymmetry transformation rules of the large vector and deformed linear multiplets have been altered.

Of course, it is easily seen that if the large vector multiplet M is restricted to be chiral, $\bar{\boldsymbol{\nabla}}_i^{\dot\alpha}M = 0$, then it reduces to the usual vector multiplet.[14] Similarly, the conditions on the deformed linear multiplet (4.23) reduce in this case to the usual constraints (3.22) for a linear multiplet.

5 Applications and discussion

Until now we have made use of the superfield M with little comment as to its physical content. It should be apparent that relative to the usual vector multiplet $\mathcal{Z}$, the multiplet M is quite enormous; and because it possesses nontrivial dilatation and $\mathrm{U}(1)_\mathrm{R}$ weights, we cannot consistently eliminate either its modulus or its phase. Nevertheless, there are several ways we might attempt to reduce it.

The simplest choice is (of course) to take M to be independent of the central charge, which amounts to choosing $M = \mathcal{Z}$ for some vector multiplet and reducing all of the structure in section 4 to that of section 3. A less trivial alternative is merely to isolate the x^5-dependence into either the modulus or the phase of M. The first choice is to take

$$\partial_5(M\bar{M}) \neq 0 \ , \qquad \partial_5(M/\bar{M}) = 0 \ . \tag{5.1}$$

Examining the Bianchi identity (4.15), we see that it is solved by

$$M/\bar{M} = \Phi/\bar\Phi \ , \tag{5.2}$$

[14]The chirality condition on M actually implies that M is independent of the central charge from consistency of the anticommutator $\{\bar{\boldsymbol{\nabla}}_i^{\dot\alpha}, \bar{\boldsymbol{\nabla}}_j^{\dot\beta}\}M = 0$.

for some chiral superfield Φ. But (4.16) then gives $\nabla^{ij}\Phi = \bar{\nabla}^{ij}\bar{\Phi}$ and so Φ is a reduced chiral multiplet, $\Phi = \mathcal{Z}$. The Bianchi identities then tell us *absolutely nothing* about the modulus of M.

Now consider the second choice,

$$\partial_5(M\bar{M}) = 0 \ , \qquad \partial_5(M/\bar{M}) \neq 0 \ . \tag{5.3}$$

The Bianchi identities tell us little about $M\bar{M}$, which is some real superfield with dilatation weight two. But because $M\bar{M}$ is x^5-independent, we may treat it as a conformal supergravity compensator. In this light, the most natural choice would seem to be $M\bar{M} = \mathcal{Z}\bar{\mathcal{Z}}$ for a vector multiplet $\mathcal{Z}$. Making this choice for the modulus of M, we identify the phase by setting

$$M = -\mathrm{i}\mathcal{Z}\mathrm{e}^{-\mathrm{i}L} \tag{5.4}$$

for some real superfield L that depends on the central charge.[15] The Bianchi identities (4.15) and (4.16) then become

$$\boldsymbol{\nabla}_\alpha^{(i}\bar{\boldsymbol{\nabla}}_{\dot{\beta}}^{j)} L = 0 \ , \qquad \mathrm{e}^{\mathrm{i}L}\boldsymbol{\nabla}^{ij}(\mathcal{Z}\mathrm{e}^{-2\mathrm{i}L}) = -\mathrm{e}^{-\mathrm{i}L}\bar{\boldsymbol{\nabla}}^{ij}(\bar{\mathcal{Z}}\mathrm{e}^{2\mathrm{i}L}) \ . \tag{5.5}$$

These equations are (some of) the constraints that define the variant vector-tensor multiplet [45].

The variant vector-tensor multiplet has been defined recently in supergravity [45]. It is a generalization (both to supergravity and with more general couplings to vector multiplets) of a multiplet introduced first by Theis [29, 30]. The simplest version involves introducing the additional constraint [45]

$$\mathrm{e}^{-\mathrm{i}L}\boldsymbol{\nabla}^{ij}(\mathcal{Z}\mathrm{e}^{2\mathrm{i}L}) = -\mathrm{e}^{\mathrm{i}L}\bar{\boldsymbol{\nabla}}^{ij}(\bar{\mathcal{Z}}\mathrm{e}^{-2\mathrm{i}L}) \ . \tag{5.6}$$

The superfield L obeying (5.5) and (5.6) describes the variant vector-tensor multiplet. Its Lagrangian is constructed from a generalized linear multiplet $\mathcal{L}^{ij}$ given by [45]

$$\mathcal{L}^{ij} = \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}L}\boldsymbol{\nabla}^{ij}(\mathcal{Z}L\mathrm{e}^{2\mathrm{i}L}) - \mathrm{e}^{\mathrm{i}L}\mathcal{Z}\boldsymbol{\nabla}^{\alpha i}L\boldsymbol{\nabla}_\alpha^j L - \frac{1}{4}\mathrm{e}^{\mathrm{i}L}\boldsymbol{\nabla}^{ij}\mathcal{Z} + \text{c.c.} \tag{5.7}$$

which can be shown to obey the constraints

$$0 = \mathrm{e}^{-\mathrm{i}L}\boldsymbol{\nabla}_\alpha^{(i}(\mathrm{e}^{\mathrm{i}L}\mathcal{L}^{jk)}) = \widetilde{\boldsymbol{\nabla}}_\alpha^{(i}\mathcal{L}^{jk)} \ , \qquad 0 = \mathrm{e}^{\mathrm{i}L}\bar{\boldsymbol{\nabla}}_{\dot{\alpha}}^{(i}(\mathrm{e}^{-\mathrm{i}L}\mathcal{L}^{jk)}) = \widetilde{\bar{\boldsymbol{\nabla}}}_{\dot{\alpha}}^{(i}\mathcal{L}^{jk)} \ , \tag{5.8}$$

We refer the reader to [45] for a full discussion.

[15] The overall choice of phase of M can be changed by redefining L. The choice made here matches that used in [45].

An alternative possibility is to use the variant vector-tensor multiplet (or some other central charge multiplet M) to construct a new action involving a *massless* Fayet-Sohnius hypermultiplet. Let us suppose q_i is a superfield obeying the constraints

$$\boldsymbol{\nabla}_\alpha^{(i} q^{j)} = \bar{\boldsymbol{\nabla}}_{\dot\beta}^{(i} q^{j)} = 0 \tag{5.9}$$

and similarly for its conjugate $\bar{q}^i := (q_i)^*$. We can introduce a composite variant linear multiplet

$$\mathcal{L}^{ij} = \frac{1}{2} \bar{q}^{(i} \overleftrightarrow{\Delta} q^{j)} \ . \tag{5.10}$$

By construction, $\mathcal{L}^{ij}$ obeys

$$\widetilde{\boldsymbol{\nabla}}_\alpha^{(i} \mathcal{L}^{jk)} = \widetilde{\boldsymbol{\nabla}}_{\dot\beta}^{(i} \mathcal{L}^{jk)} = 0 \tag{5.11}$$

and its action may be constructed directly using (4.34). As with the usual Fayet-Sohnius hypermultiplet, this multiplet has the equation of motion $\Delta q_i = 0$ and so the on-shell hypermultiplet decouples from the large vector multiplet.[16]

It has recently been shown at the component level [51] that 5D $\mathcal{N} = 1$ conformal supergravity can be dimensionally reduced off-shell to 4D $\mathcal{N} = 2$ conformal supergravity coupled to a vector multiplet. One expects that this component construction can be repeated at the superfield level and thereby connect 5D $\mathcal{N} = 1$ superspace directly to the central charge superspace we considered in section 3. A natural question to ask is whether the more general central charge structure described in section 4, involving a large vector multiplet, has any significance from a 5D point of view. In particular, can one construct actions in 5D which preferentially reduce in 4D so that the central charge multiplet retains x^5-dependence?

We are aware of no examples, but there is one interesting possibility. It was pointed out in [46] that the nonlinear vector-tensor multiplet has a simple 5D origin, at least in flat superspace. It was noted recently by two of us (DB and JN) [39] that the generalization of the vector-tensor multiplet to conformal supergravity [20, 21] can also be interpreted as arising from a certain 5D action. In both of these situations, the central charge is gauged by the usual vector multiplet. However, the variant vector-tensor multiplet [29, 30, 45] *itself* gauges the central charge, and so the central charge

[16] In the case that the central charge is gauged using a standard vector multiplet, the hypermultiplet Lagrangian can include a mass term, $\mathcal{L}^{ij} = \frac{1}{2} \bar{q}^{(i} \overleftrightarrow{\Delta} q^{j)} + \mathrm{i}\, m\, \bar{q}^{(i} q^{j)}$, with m a real mass parameter. No mass term is allowed if a large vector multiplet is used.

multiplet must retain x^5-dependence. Should this variant VT multiplet possess a 5D origin, it would provide just such an example.

We conclude this paper with a final comment. Within the superconformal tensor calculus, the two main types of locally supersymmetric actions are: (i) the chiral action; and (ii) the linear multiplet action. The ectoplasm construction for the chiral action was given in [53]. The case of the linear multiplet action has been worked out in the present paper.

Acknowledgements:
The work of DB and SMK is supported in part by the Australian Research Council. The work of JN is supported by an Australian Postgraduate Award.

A Conformal supergravity in 4D $\mathcal{N} = 2$ superspace

In this appendix, we briefly summarize the algebra of $\mathcal{N} = 2$ conformal superspace [36] as reformulated in [39].

A.1 $\mathcal{N} = 2$ conformal superspace

The covariant derivative $\nabla_A = (\nabla_a, \nabla_\alpha^i, \bar{\nabla}^{\dot\alpha}_i)$ is given by

$$\begin{aligned}\nabla_A &= E_A + \frac{1}{2}\Omega_A{}^{ab}M_{ab} + \Phi_A{}^{ij}J_{ij} + \mathrm{i}\Phi_A Y + B_A\mathbb{D} + \mathfrak{F}_A{}^B K_B \\ &= E_A + \Omega_A{}^{\beta\gamma}M_{\beta\gamma} + \bar{\Omega}_A{}^{\dot\beta\dot\gamma}\bar{M}_{\dot\beta\dot\gamma} + \Phi_A{}^{ij}J_{ij} + \mathrm{i}\Phi_A Y + B_A\mathbb{D} + \mathfrak{F}_A{}^B K_B \ . \end{aligned} \tag{A.1}$$

Here $E_A = E_A{}^M\partial_M$ is the supervielbein, $\Omega_A{}^{ab}$ is the spin connection, and $\Phi_A{}^{ij}$ and Φ_A are the $\mathrm{SU}(2)_\mathrm{R}$ and $\mathrm{U}(1)_\mathrm{R}$ connections, respectively. In addition, we have a dilatation connection B_A and a special superconformal connection $\mathfrak{F}_A{}^B$.

The Lorentz generators M_{ab} obey

$$[M_{ab}, \nabla_c] = 2\eta_{c[a}\nabla_{b]} \ , \quad [M_{ab}, \nabla_\alpha^i] = (\sigma_{ab})_\alpha{}^\beta\nabla_\beta^i \ , \quad [M_{ab}, \bar{\nabla}^{\dot\alpha}_i] = (\tilde{\sigma}_{ab})^{\dot\alpha}{}_{\dot\beta}\bar{\nabla}^{\dot\beta}_i \ . \tag{A.2}$$

As usual, they may be decomposed into left-handed and right-handed generators

$$M_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta}M_{ab} \ , \qquad \bar{M}_{\dot\alpha\dot\beta} = -\frac{1}{2}(\tilde{\sigma}^{ab})_{\dot\alpha\dot\beta}M_{ab} \ , \tag{A.3a}$$

$$M_{ab} = (\sigma^{ab})_{\alpha\beta}M_{\alpha\beta} - (\tilde{\sigma}^{ab})_{\dot\alpha\dot\beta}\bar{M}_{\dot\alpha\dot\beta} \ , \tag{A.3b}$$

which act only on undotted and dotted indices, respectively

$$[M_{\alpha\beta}, \nabla^i_\gamma] = \varepsilon_{\gamma(\alpha}\nabla^i_{\beta)} \ , \qquad [\bar{M}_{\dot\alpha\dot\beta}, \bar\nabla_{\dot\gamma i}] = \varepsilon_{\dot\gamma(\dot\alpha}\bar\nabla_{\dot\beta)i} \ . \tag{A.4}$$

The $\mathrm{SU}(2)_\mathrm{R}$, $\mathrm{U}(1)_\mathrm{R}$ and dilatation generators obey

$$\begin{aligned} &[J_{ij}, \nabla^k_\alpha] = -\delta^k_{(i}\nabla_{\alpha j)} \ , \quad [J_{ij}, \bar\nabla^{\dot\alpha}_k] = -\varepsilon_{k(i}\bar\nabla^{\dot\alpha}_{j)} \ , \\ &[Y, \nabla^i_\alpha] = \nabla^i_\alpha \ , \quad [Y, \bar\nabla^{\dot\alpha}_i] = -\bar\nabla^{\dot\alpha}_i \ , \\ &[\mathbb{D}, \nabla_a] = \nabla_a \ , \quad [\mathbb{D}, \nabla^i_\alpha] = \frac{1}{2}\nabla^i_\alpha \ , \quad [\mathbb{D}, \bar\nabla^{\dot\alpha}_i] = \frac{1}{2}\bar\nabla^{\dot\alpha}_i \ . \end{aligned} \tag{A.5}$$

The special superconformal generators $K^A = (K^a, S^\alpha_i, \bar{S}^i_{\dot\alpha})$ transform in the obvious way under Lorentz and $\mathrm{SU}(2)_\mathrm{R}$ generators,

$$\begin{aligned} &[M_{ab}, K_c] = 2\eta_{c[a}K_{b]} \ , \quad [M_{ab}, S^\gamma_i] = -(\sigma_{ab})_\beta{}^\gamma S^\beta_i \ , \quad [M_{ab}, \bar{S}^i_{\dot\gamma}] = -(\sigma_{ab})^{\dot\beta}{}_{\dot\gamma}\bar{S}^i_{\dot\beta} \ , \\ &[J_{ij}, S^\gamma_k] = -\varepsilon_{k(i}S^\gamma_{j)} \ , \quad [J_{ij}, \bar{S}^k_{\dot\gamma}] = -\delta^k_{(i}\bar{S}_{\dot\gamma j)} \ , \end{aligned} \tag{A.6}$$

and carry opposite $\mathrm{U}(1)_\mathrm{R}$ and dilatation weight to ∇_A:

$$\begin{aligned} &[Y, S^\alpha_i] = -S^\alpha_i \ , \quad [Y, \bar{S}^i_{\dot\alpha}] = \bar{S}^i_{\dot\alpha} \ , \\ &[\mathbb{D}, K_a] = -K_a \ , \quad [\mathbb{D}, S^\alpha_i] = -\frac{1}{2}S^\alpha_i \ , \quad [\mathbb{D}, \bar{S}^i_{\dot\alpha}] = -\frac{1}{2}\bar{S}^i_{\dot\alpha} \ . \end{aligned} \tag{A.7}$$

Among themselves, the generators K^A obey the algebra

$$\{S^\alpha_i, \bar{S}^j_{\dot\alpha}\} = 2\mathrm{i}\delta^j_i(\sigma^a)^\alpha{}_{\dot\alpha}K_a \ . \tag{A.8}$$

with all the other (anti-)commutators vanishing.

Finally, the algebra of K^A with ∇_B is given by

$$\begin{aligned} &[K^a, \nabla_b] = 2\delta^a_b\mathbb{D} + 2M^a{}_b \ , \\ &\{S^\alpha_i, \nabla^j_\beta\} = 2\delta^j_i\delta^\alpha_\beta\mathbb{D} - 4\delta^j_i M^\alpha{}_\beta - \delta^j_i\delta^\alpha_\beta Y + 4\delta^\alpha_\beta J_i{}^j \ , \\ &\{\bar{S}^i_{\dot\alpha}, \bar\nabla^{\dot\beta}_j\} = 2\delta^i_j\delta^{\dot\beta}_{\dot\alpha}\mathbb{D} + 4\delta^i_j\bar{M}_{\dot\alpha}{}^{\dot\beta} + \delta^i_j\delta^{\dot\beta}_{\dot\alpha}Y - 4\delta^{\dot\beta}_{\dot\alpha}J^i{}_j \ , \\ &[K^a, \nabla^j_\beta] = -\mathrm{i}(\sigma^a)_\beta{}^{\dot\beta}\bar{S}^j_{\dot\beta} \ , \quad [K^a, \bar\nabla^{\dot\beta}_j] = -\mathrm{i}(\sigma^a)^{\dot\beta}{}_\beta S^\beta_j \ , \\ &[S^\alpha_i, \nabla_b] = \mathrm{i}(\sigma_b)^\alpha{}_{\dot\beta}\bar\nabla^{\dot\beta}_i \ , \quad [\bar{S}^i_{\dot\alpha}, \nabla_b] = \mathrm{i}(\sigma_b)_{\dot\alpha}{}^\beta\nabla^i_\beta \ , \end{aligned} \tag{A.9}$$

where all other (anti-)commutations vanish.

The algebra of covariant derivatives has the form

$$\begin{aligned} [\nabla_A, \nabla_B\} = {} & T_{AB}{}^C\nabla_C + \frac{1}{2}R_{AB}{}^{cd}M_{cd} + R_{AB}{}^{kl}J_{kl} \\ & + \mathrm{i}R_{AB}(Y)Y + R_{AB}(\mathbb{D})\mathbb{D} + R_{AB}{}^C K_C \ . \end{aligned} \tag{A.10}$$

We impose constraints on the curvatures appearing on the right-hand side to reproduce the component structure of conformal supergravity [36]. The resulting algebra is given by

$$\{\nabla^i_\alpha, \nabla^j_\beta\} = 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{W}_{\dot\gamma\dot\delta}\bar{M}^{\dot\gamma\dot\delta} + \frac{1}{2}\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar\nabla_{\dot\gamma k}\bar{W}^{\dot\gamma\dot\delta}\bar{S}^k_{\dot\delta} - \frac{1}{2}\varepsilon^{ij}\varepsilon_{\alpha\beta}\nabla_{\gamma\dot\delta}\bar{W}^{\dot\delta}{}_{\dot\gamma}K^{\gamma\dot\gamma} \ , \tag{A.11a}$$

$$\{\bar\nabla^{\dot\alpha}_i, \bar\nabla^{\dot\beta}_j\} = -2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}W^{\gamma\delta}M_{\gamma\delta} + \frac{1}{2}\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\nabla^{\gamma k}W_{\gamma\delta}S^\delta_k - \frac{1}{2}\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\nabla^{\gamma\dot\gamma}W_\gamma{}^\delta K_{\delta\dot\gamma} \ , \tag{A.11b}$$

$$\{\nabla^i_\alpha, \bar\nabla^{\dot\beta}_j\} = -2\mathrm{i}\delta^i_j\nabla_\alpha{}^{\dot\beta} \ , \tag{A.11c}$$

$$\begin{aligned}[\nabla_{\alpha\dot\alpha}, \nabla^i_\beta] = {}& -\mathrm{i}\varepsilon_{\alpha\beta}\bar{W}_{\dot\alpha\dot\beta}\bar\nabla^{\dot\beta i} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\bar\nabla^{\dot\beta i}\bar{W}_{\dot\alpha\dot\beta}\mathbb{D} - \frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\bar\nabla^{\dot\beta i}\bar{W}_{\dot\alpha\dot\beta}Y + \mathrm{i}\varepsilon_{\alpha\beta}\bar\nabla^{\dot\beta}_j\bar{W}_{\dot\alpha\dot\beta}J^{ij} \\ & - \mathrm{i}\varepsilon_{\alpha\beta}\bar\nabla^i_{\dot\beta}\bar{W}_{\dot\gamma\dot\alpha}\bar{M}^{\dot\beta\dot\gamma} - \frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\bar\nabla^i_{\dot\alpha}\bar\nabla^{\dot\beta}_k\bar{W}_{\dot\beta\dot\gamma}\bar{S}^{\dot\gamma k} + \frac{1}{2}\varepsilon_{\alpha\beta}\nabla^{\gamma\dot\beta}\bar{W}_{\dot\alpha\dot\beta}S^i_\gamma \\ & + \frac{\mathrm{i}}{4}\varepsilon_{\alpha\beta}\bar\nabla^i_{\dot\alpha}\nabla^\gamma{}_{\dot\gamma}\bar{W}^{\dot\gamma\dot\beta}K_{\gamma\dot\beta} \ ,\end{aligned} \tag{A.11d}$$

$$\begin{aligned}[\nabla_{\alpha\dot\alpha}, \bar\nabla^{\dot\beta}_i] = {}& \mathrm{i}\delta^{\dot\beta}_{\dot\alpha}W_{\alpha\beta}\nabla^\beta_i + \frac{\mathrm{i}}{2}\delta^{\dot\beta}_{\dot\alpha}\nabla^\beta_i W_{\alpha\beta}\mathbb{D} - \frac{\mathrm{i}}{4}\delta^{\dot\beta}_{\dot\alpha}\nabla^\beta_i W_{\alpha\beta}Y + \mathrm{i}\delta^{\dot\beta}_{\dot\alpha}\nabla^{\beta j}W_{\alpha\beta}J_{ij} \\ & + \mathrm{i}\delta^{\dot\beta}_{\dot\alpha}\nabla^\beta_i W^\gamma{}_\alpha M_{\beta\gamma} + \frac{\mathrm{i}}{4}\delta^{\dot\beta}_{\dot\alpha}\nabla_{\alpha i}\nabla^{\beta j}W_\beta{}^\gamma S_{\gamma j} - \frac{1}{2}\delta^{\dot\beta}_{\dot\alpha}\nabla^\beta{}_{\dot\gamma}W_{\alpha\beta}\bar{S}^{\dot\gamma}_i \\ & + \frac{\mathrm{i}}{4}\delta^{\dot\beta}_{\dot\alpha}\nabla_{\alpha i}\nabla^\gamma{}_{\dot\gamma}W_{\beta\gamma}K^{\beta\dot\gamma} \ .\end{aligned} \tag{A.11e}$$

We have not given the algebra of two vector covariant derivatives since it may straightforwardly be derived as a consequence of the above (anti)-commutators. The result is given in [36].

The curvatures are characterized by a single complex superfield $W_{\alpha\beta}$, which is the superconformal Weyl tensor. It is symmetric ($W_{\alpha\beta} = W_{\beta\alpha}$), superconformally primary ($K_A W_{\alpha\beta} = 0$), chiral ($\bar\nabla^{\dot\alpha}_i W_{\beta\gamma} = 0$), and obeys the Bianchi identity

$$\nabla_{\alpha\beta}W^{\alpha\beta} = \bar\nabla^{\dot\alpha\dot\beta}\bar{W}_{\dot\alpha\dot\beta} \ , \tag{A.12}$$

where we introduce the notation

$$\nabla_{\alpha\beta} := \nabla^k_{(\alpha}\nabla_{\beta)k} \ , \quad \bar\nabla^{\dot\alpha\dot\beta} := \bar\nabla^{(\dot\alpha}_k\bar\nabla^{\dot\beta)k} \ . \tag{A.13}$$

A.2 Conformal supergravity and central charge

As in, *e.g.* [39] we can introduce a central charge gauged by an off-shell vector multiplet. First, we introduce a modified covariant derivative:

$$\boldsymbol{\nabla}_A := \nabla_A + V_A\Delta \ , \tag{A.14}$$

where $V_A(z)$ is the gauge connection and Δ is a real central charge. We assume that the central charge commutes with the modified covariant derivative, $[\Delta, \boldsymbol{\nabla}_A] = 0$,

which allows us to treat it completely analogously to a U(1) generator as far as the algebra is concerned.

The curvature tensors are given by

$$
\begin{aligned}
[\boldsymbol{\nabla}_A, \boldsymbol{\nabla}_B\} = T_{AB}{}^C \boldsymbol{\nabla}_C + F_{AB}\Delta + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^{kl} J_{kl} \\
+ \mathrm{i} R_{AB}(Y) Y + R_{AB}(\mathbb{D})\mathbb{D} + R_{AB}{}^C K_C \ .
\end{aligned}
\tag{A.15}
$$

We then impose the constraints for the vector multiplet, using $\mathcal{Z}$ to denote the corresponding abelian field strength,

$$
F_{\alpha\beta}^{i\,j} = 2\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{\mathcal{Z}} \ , \quad F^{\dot\alpha\dot\beta}_{i\ j} = -2\varepsilon^{\dot\alpha\dot\beta}\varepsilon_{ij}\mathcal{Z} \ , \qquad F_{\alpha j}^{i\,\dot\beta} = 0 \ , \tag{A.16a}
$$

$$
F_{a\beta}^{\ j} = -\frac{\mathrm{i}}{2}(\sigma_a)_\beta{}^{\dot\gamma}\bar{\boldsymbol{\nabla}}^j_{\dot\gamma}\bar{\mathcal{Z}} \ , \qquad F_{a j}^{\ \dot\beta} = \frac{\mathrm{i}}{2}(\sigma_a)_\gamma{}^{\dot\beta}\boldsymbol{\nabla}^\gamma_j\mathcal{Z} \ , \tag{A.16b}
$$

$$
F_{ab} = \frac{1}{8}(\sigma_{ab})_{\alpha\beta}(\boldsymbol{\nabla}^{\alpha\beta}\mathcal{Z} + 4W^{\alpha\beta}\bar{\mathcal{Z}}) - \frac{1}{8}(\tilde\sigma_{ab})_{\dot\alpha\dot\beta}(\bar{\boldsymbol{\nabla}}^{\dot\alpha\dot\beta}\bar{\mathcal{Z}} + 4\bar W^{\dot\alpha\dot\beta}\mathcal{Z}) \ , \tag{A.16c}
$$

where $\mathcal{Z}$ is a reduced chiral primary superfield with dimension 1 and U(1) weight -2

$$
\begin{aligned}
&K_A\mathcal{Z} = 0 \ , \quad \mathbb{D}\mathcal{Z} = \mathcal{Z} \ , \quad Y\mathcal{Z} = -2\mathcal{Z} \ , \\
&\bar\nabla^i_{\dot\alpha}\mathcal{Z} = 0 \ , \qquad \nabla^{ij}\mathcal{Z} = \bar\nabla^{ij}\bar{\mathcal{Z}} \ .
\end{aligned}
\tag{A.17}
$$

Our normalization for $\mathcal{Z}$ has been chosen so that in the flat limit taking $\mathcal{Z} \to 1$ allows the identification of the central charge curvature F with the five-dimensional torsion tensor T^5.

B $\mathcal{N} = 1$ BF coupling via ectoplasm

In this appendix, we briefly discuss how to use the ectoplasm method to construct the BF action in $\mathcal{N} = 1$ conformal supergravity corresponding to

$$
S_{\mathrm{BF}} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, E\, LV \tag{B.1}
$$

where V is a vector multiplet prepotential and L is a real linear multiplet obeying

$$
\nabla^2 L = \bar\nabla^2 L = 0 \ . \tag{B.2}
$$

We work in $\mathcal{N} = 1$ conformal superspace [54] but use the Lorentz conventions consistent with the rest of the paper.[17]

[17]The standard formulation of $\mathcal{N} = 1$ conformal supergravity [34] can be obtained from this formulation by an appropriate gauge-fixing [54, 55].

Let Σ be a superspace four-form obeying the equation

$$\mathrm{d}\Sigma + F \wedge H = 0 \tag{B.3}$$

for closed two-form F and closed three-form H. The field strength F is given by

$$F_{\hat\beta\hat\alpha} = 0 \ , \qquad F_{\beta a} = (\sigma^a)_{\beta\dot\beta} \bar W^{\dot\beta} \ , \qquad F_{\dot\beta a} = (\sigma^a)_{\beta\dot\beta} W^\beta \ , \tag{B.4a}$$

$$F_{ba} = \frac{\mathrm{i}}{2}(\sigma_{ba})^{\alpha\beta}\nabla_\beta W_\alpha - \frac{\mathrm{i}}{2}(\tilde\sigma_{ba})_{\dot\alpha\dot\beta}\bar\nabla^{\dot\beta}\bar W^{\dot\alpha} \ , \tag{B.4b}$$

where W_α is a reduced chiral spinor obeying $\nabla^\alpha W_\alpha = \bar\nabla_{\dot\alpha}\bar W^{\dot\alpha}$. The three-form H is given by

$$H_{\hat\gamma\hat\beta\hat\alpha} = 0 \ , \qquad H_{\gamma\beta a} = H_{\dot\gamma\dot\beta a} = 0 \ , \qquad H_{\gamma\dot\beta a} = 2\mathrm{i}(\sigma_a)_{\gamma\dot\beta} L \ , \tag{B.5a}$$

$$H_{\gamma ba} = -2(\sigma_{ba})_\gamma{}^\delta \nabla_\delta L \ , \qquad H_{\dot\gamma ba} = -2(\tilde\sigma_{ba})_{\dot\gamma\dot\delta}\bar\nabla^{\dot\delta} L \ , \tag{B.5b}$$

$$H_{cba} = -\frac{1}{4}\varepsilon_{dcba}(\tilde\sigma^d)^{\dot\alpha\alpha}[\nabla_\alpha, \bar\nabla_{\dot\alpha}]L \ . \tag{B.5c}$$

If we constrain certain components of the four-form Σ to be zero, we find

$$\Sigma_{\hat\delta\hat\gamma\hat\beta A} = 0 \ , \tag{B.6a}$$

$$\Sigma_{\delta\gamma ba} = \frac{1}{16}(\sigma_{ba})_{\delta\gamma}\bar Y \ , \qquad \Sigma^{\dot\delta\dot\gamma}{}_{ba} = \frac{1}{16}(\tilde\sigma_{ba})^{\dot\delta\dot\gamma} Y \ , \qquad \Sigma_{\delta\dot\gamma ba} = 0 \ , \tag{B.6b}$$

$$\Sigma_{\delta cba} = -\frac{1}{16}\varepsilon_{dcba}(\sigma^d)_{\delta\dot\delta}(\bar\nabla^{\dot\delta}\bar Y - 16\mathrm{i}\,\bar W^{\dot\delta} L) \ , \tag{B.6c}$$

$$\Sigma^{\dot\delta}{}_{cba} = +\frac{1}{16}\varepsilon_{dcba}(\tilde\sigma^d)^{\dot\delta\delta}(\nabla_\delta Y + 16\mathrm{i}\, W_\delta L) \ , \tag{B.6d}$$

$$\Sigma_{dcba} = \varepsilon_{dcba}\Big(\frac{\mathrm{i}}{64}(\nabla^2 Y - \bar\nabla^2\bar Y) - W^\alpha\nabla_\alpha L - \bar W_{\dot\alpha}\bar\nabla^{\dot\alpha} L - \frac{1}{2}L\nabla^\alpha W_\alpha\Big) \ , \tag{B.6e}$$

where Y is a chiral superfield. The Y-dependent pieces of Σ correspond to a closed four-form Σ_c, which was constructed in [41] (see also [53]) and generates a chiral action in $\mathcal{N} = 1$ supergravity. This closed four-form Σ_c coincides with the field strength of a gauge three-form in the case that $Y = \bar\nabla^2 X$ with $X = \bar X$ [48, 56].

Now we introduce a closed four-form J

$$J := \Sigma + A \wedge H \ , \qquad \mathrm{d}J = \mathrm{d}\Sigma - \mathrm{d}A \wedge H = 0 \ . \tag{B.7}$$

The four-form J_{mnpq},

$$J_{mnpq} = \Sigma_{mnpq} - 4A_{[m}H_{npq]} \ , \tag{B.8}$$

then gives a supersymmetric four-form action. Neglecting gravitinos, we find

$$\begin{aligned}\frac{1}{4!}\varepsilon^{mnpq}J_{mnpq} = &\frac{\mathrm{i}}{64}(\nabla^2 Y - \bar\nabla^2\bar Y) - W^\alpha\nabla_\alpha L - \bar W_{\dot\alpha}\bar\nabla^{\dot\alpha} L - \frac{1}{2}L\nabla^\alpha W_\alpha \\ &- \frac{1}{4}A^{\dot\alpha\alpha}[\nabla_\alpha, \bar\nabla_{\dot\alpha}]L + \mathcal{O}(\psi) \ .\end{aligned} \tag{B.9}$$

If we further constrain $Y = 0$, this is exactly the component Lagrangian for the action (B.1) under the identifications $W_\alpha = -\frac{1}{8}\bar{\nabla}^2\nabla_\alpha V$ and $A_{\alpha\dot{\alpha}} = -\frac{1}{4}[\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}]V$.

References

[1] M. F. Sohnius, "Supersymmetry and central charges," Nucl. Phys. B **138**, 109 (1978).

[2] R. Grimm, M. Sohnius and J. Wess, "Extended supersymmetry and gauge theories," Nucl. Phys. B **133**, 275 (1978).

[3] P. Breitenlohner and M. F. Sohnius, "Superfields, auxiliary fields, and tensor calculus for N=2 extended supergravity," Nucl. Phys. B **165**, 483 (1980).

[4] S. Ferrara, J. Wess and B. Zumino, "Supergauge multiplets and superfields," Phys. Lett. **B51**, 239 (1974).

[5] W. Siegel, "Gauge spinor superfield as a scalar multiplet," Phys. Lett. B **85**, 333 (1979).

[6] M. F. Sohnius, K. S. Stelle and P. C. West, "Representations of extended supersymmetry," in *Superspace and Supergravity*, S. W. Hawking and M. Roček (Eds.), Cambridge University Press, Cambridge, 1981, p. 283.

[7] J. Wess, "Supersymmetry and internal symmetry," Acta Phys. Austriaca **41** (1975) 409.

[8] B. de Wit, J. W. van Holten and A. Van Proeyen, "Central charges and conformal supergravity," Phys. Lett. B **95**, 51 (1980).

[9] B. de Wit, J. W. van Holten and A. Van Proeyen, "Transformation rules of N=2 supergravity multiplets," Nucl. Phys. B **167**, 186 (1980).

[10] E. Bergshoeff, M. de Roo and B. de Wit, "Extended conformal supergravity," Nucl. Phys. B **182**, 173 (1981).

[11] B. de Wit, J. W. van Holten and A. Van Proeyen, "Structure of N=2 supergravity," Nucl. Phys. B **184**, 77 (1981)h [Erratum-ibid. B **222**, 516 (1983)].

[12] A. S. Galperin, E. A. Ivanov, S. N. Kalitsyn, V. Ogievetsky, E. Sokatchev, "Unconstrained N=2 matter, Yang-Mills and supergravity theories in harmonic superspace," Class. Quant. Grav. **1**, 469 (1984).

[13] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, *Harmonic Superspace*, Cambridge University Press, Cambridge, 2001.

[14] A. Karlhede, U. Lindström and M. Roček, "Self-interacting tensor multiplets in N=2 superspace," Phys. Lett. B **147**, 297 (1984).

[15] U. Lindström and M. Roček, "New hyperkähler metrics and new supermultiplets," Commun. Math. Phys. **115**, 21 (1988); "N=2 super Yang-Mills theory in projective superspace," Commun. Math. Phys. **128**, 191 (1990).

[16] A. A. Rosly, “Super Yang-Mills constraints as integrability conditions,” in Proceedings of the International Seminar *Group Theoretical Methods in Physics* (Zvenigorod, USSR, 1982), M. A. Markov (Ed.), Nauka, Moscow, 1983, Vol. 1, p. 263 (in Russian); English translation: in *Group Theoretical Methods in Physics*,” M. A. Markov, V. I. Man’ko and A. E. Shabad (Eds.), Harwood Academic Publishers, London, Vol. 3, 1987, p. 587.

[17] M. Sohnius, K. S. Stelle and P. C. West, “Off-mass-shell formulation of extended supersymmetric gauge theories,” Phys. Lett. B **92**, 123 (1980); “Dimensional reduction by Legendre transformation generates off-shell supersymmetric Yang-Mills theories,” Nucl. Phys. B **173**, 127 (1980).

[18] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, “Perturbative couplings of vector multiplets in N=2 heterotic string vacua,” Nucl. Phys. B **451**, 53 (1995) [arXiv:hep-th/9504006].

[19] P. Fayet, “Fermi-Bose hypersymmetry,” Nucl. Phys. B **113**, 135 (1976).

[20] P. Claus, B. de Wit, M. Faux, B. Kleijn, R. Siebelink and P. Termonia, “The vector-tensor supermultiplet with gauged central charge,” Phys. Lett. B **373**, 81 (1996) [arXiv:hep-th/9512143].

[21] P. Claus, P. Termonia, B. de Wit and M. Faux, “Chern-Simons couplings and inequivalent vector-tensor multiplets,” Nucl. Phys. B **491**, 201 (1997) [arXiv:hep-th/9612203].

[22] A. Hindawi, B. A. Ovrut and D. Waldram, “Vector-tensor multiplet in N=2 superspace with central charge,” Phys. Lett. B **392**, 85 (1997) [arXiv:hep-th/9609016].

[23] R. Grimm, M. Hasler and C. Herrmann, “The N=2 vector-tensor multiplet, central charge superspace, and Chern-Simons couplings,” Int. J. Mod. Phys. A **13**, 1805 (1998) [arXiv:hep-th/9706108].

[24] N. Dragon, S. M. Kuzenko and U. Theis, “The vector-tensor multiplet in harmonic superspace,” Eur. Phys. J. C **4**, 717 (1998) [arXiv:hep-th/9706169].

[25] I. Buchbinder, A. Hindawi and B. A. Ovrut, “A two-form formulation of the vector-tensor multiplet in central charge superspace,” Phys. Lett. B **413**, 79 (1997) [arXiv:hep-th/9706216].

[26] N. Dragon and S. M. Kuzenko, “Self-interacting vector-tensor multiplet,” Phys. Lett. B **420**, 64 (1998) [arXiv:hep-th/9709088].

[27] E. Ivanov and E. Sokatchev, “On nonlinear superfield versions of the vector tensor multiplet,” Phys. Lett. B **429**, 35 (1998) [arXiv:hep-th/9711038].

[28] N. Dragon, E. Ivanov, S. Kuzenko, E. Sokatchev and U. Theis, “N=2 rigid supersymmetry with gauged central charge,” Nucl. Phys. B **538**, 411 (1999) [arXiv:hep-th/9805152].

[29] U. Theis, “New N=2 supersymmetric vector tensor interaction,” Phys. Lett. B **486**, 443 (2000) [arXiv:hep-th/0005044].

[30] U. Theis, “Nonlinear vector tensor multiplets revisited,” Nucl. Phys. B **602**, 367 (2001) [arXiv:hep-th/0012096].

[31] S. M. Kuzenko and S. Theisen, “Correlation functions of conserved currents in N = 2 superconformal theory,” Class. Quant. Grav. **17**, 665 (2000) [hep-th/9907107].

[32] A. S. Galperin, N. A. Ky and E. Sokatchev, “N=2 supergravity in superspace: Solution to the constraints,” Class. Quant. Grav. **4**, 1235 (1987).

[33] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. Sokatchev, “N=2 supergravity in superspace: Different versions and matter couplings,” Class. Quant. Grav. **4**, 1255 (1987).

[34] P. S. Howe, “Supergravity in superspace,” Nucl. Phys. B **199**, 309 (1982).

[35] S. M. Kuzenko, U. Lindström, M. Roček and G. Tartaglino-Mazzucchelli, “4D N=2 supergravity and projective superspace,” JHEP **0809**, 051 (2008) [arXiv:0805.4683 [hep-th]].

[36] D. Butter, “N=2 conformal superspace in four dimensions,” JHEP **1110**, 030 (2011). [arXiv:1103.5914 [hep-th]].

[37] S. M. Kuzenko, U. Lindström, M. Roček and G. Tartaglino-Mazzucchelli, “On conformal supergravity and projective superspace,” JHEP **0908**, 023 (2009) [arXiv:0905.0063 [hep-th]].

[38] S. M. Kuzenko and J. Novak, “Vector-tensor supermultiplets in AdS and supergravity,” JHEP **1201**, 106 (2012) [arXiv:1110.0971 [hep-th]].

[39] D. Butter and J. Novak, “Component reduction in N=2 supergravity: the vector, tensor, and vector-tensor multiplets,” JHEP **1205**, 115 (2012) [arXiv:1201.5431 [hep-th]].

[40] S. J. Gates, Jr., “Ectoplasm has no topology: The prelude,” in *Supersymmetries and Quantum Symmetries*, J. Wess and E. A. Ivanov (Eds.), Springer, Berlin, 1999, p. 46, arXiv:hep-th/9709104; “Ectoplasm has no topology,” Nucl. Phys. B **541**, 615 (1999) [arXiv:hep-th/9809056].

[41] S. J. Gates, Jr., M. T. Grisaru, M. E. Knutt-Wehlau and W. Siegel, “Component actions from curved superspace: Normal coordinates and ectoplasm,” Phys. Lett. B **421**, 203 (1998) [hep-th/9711151].

[42] T. Voronov, “Geometric integration theory on supermanifolds,” Sov. Sci. Rev. C: Math. Phys. **9**, 1 (1992).

[43] M. F. Hasler, “The three-form multiplet in N=2 superspace,” Eur. Phys. J. C **1**, 729 (1998) [hep-th/9606076].

[44] G. Girardi and R. Grimm, “N=1 supergravity: Topological classes and superspace geometry in four-dimensions,” Phys. Lett. B **260**, 365 (1991).

[45] J. Novak, “Superform formulation for vector-tensor multiplets in conformal supergravity,” JHEP **1209**, 060 (2012) [arXiv:1205.6881 [hep-th]].

[46] S. M. Kuzenko and W. D. Linch, “On five-dimensional superspaces,” JHEP **0602**, 038 (2006) [hep-th/0507176].

[47] P. S. Howe, T. G. Pugh, K. S. Stelle and C. Strickland-Constable, “Ectoplasm with an edge,” JHEP **1108**, 081 (2011) [arXiv:1104.4387 [hep-th]].

[48] S. J. Gates, Jr., “Super p-form gauge superfields,” Nucl. Phys. B **184**, 381 (1981).

[49] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, “Super-Weyl invariance in 5D supergravity,” JHEP **0804**, 032 (2008) [arXiv:0802.3953 [hep-th]].

[50] T. Kugo and K. Ohashi, “Supergravity tensor calculus in 5D from 6D,” Prog. Theor. Phys. **104**, 835 (2000) [hep-ph/0006231].

[51] N. Banerjee, B. de Wit and S. Katmadas, "The off-shell 4D/5D connection," JHEP **1203**, 061 (2012) [arXiv:1112.5371 [hep-th]].

[52] G. Akemann, R. Grimm, M. Hasler and C. Herrmann, "N=2 central charge superspace and a minimal supergravity multiplet," Class. Quant. Grav. **16**, 1617 (1999) [hep-th/9812026].

[53] S. J. Gates, Jr., S. M. Kuzenko and G. Tartaglino-Mazzucchelli, "Chiral supergravity actions and superforms," Phys. Rev. D **80**, 125015 (2009) [arXiv:0909.3918 [hep-th]].

[54] D. Butter, "N=1 conformal superspace in four dimensions," Annals Phys. **325**, 1026 (2010) [arXiv:0906.4399 [hep-th]].

[55] D. Butter and S. M. Kuzenko, "A dual formulation of supergravity-matter theories," Nucl. Phys. B **854**, 1 (2012) [arXiv:1106.3038 [hep-th]].

[56] P. Binetruy, G. Girardi and R. Grimm, "Supergravity couplings: A geometric formulation," Phys. Rept. **343**, 255 (2001) [hep-th/0005225].